1.5 Flatness and Horizon Problems
Define

$$\Omega = \frac{\rho(t)}{\rho_{crit}(t)},$$

where $\rho_{crit} = \frac{3H^2}{8\pi G}$. We have that

$$H^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2}$$

(the Friedman equation), and the Raychauduri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P).$$

Assume that $P = w\rho$, where w is a constant. We can show that

$$\dot{\Omega} = (1+3w)H\Omega(\Omega - 1).$$

If $1+3w>0$ (so far, we have considered $w=0$ for matter and $w=1/3$ for radiation), then $\Omega = 1$ is a repelling solution. So unless Ω is exactly one, the solution diverges. This is the Flatness Problem. It is a problem because Ω has been measured to be very close to 1, even after all this time – why?

In fact, it moves away from $\Omega = 1$ very quickly indeed. Since $\Omega_{tot} = \Omega_{rad} + \Omega_m + \Omega_\Lambda \approx 1$ today, this implies that it must have been extremely close to 1 at earlier times ("fine tuning").

The solution for the equation is

$$\Omega(t) = \frac{1}{1 - \left(\frac{\Omega_0 - 1}{\Omega_0}\right)a^{1+2w}}.$$

Where Ω_0 is the value for Ω today.

If $\Omega_0 = 1 + \varepsilon$, where $\varepsilon << 1$, then

$$\Omega = \frac{1}{1 - \frac{\varepsilon}{1+\varepsilon}a^{1+3w}}.$$

At Ω_{eq}, where $w=0$, this can be approximated to

$$\begin{aligned}\Omega_{eq} &\approx 1 + \varepsilon a_{eq} \\ &\approx 1 + 10^{-4}\varepsilon\end{aligned}$$

as $a_{eq} \approx 10^{-4}$. ($1 + z_{eq} = \frac{1}{a_{eq}} \approx 10^4$). Hence

$$|\Omega_{eq} - 1| = 10^{-4}|\Omega_0 - 1|$$

So Ω is closer to 1 at t_{eq} → the Flatness Problem. Ω must be fine-tuned to be very close to 1 at the time of the Big Bang.

Now consider the horizon size today compared to the horizon at $t = t_{rec}$.

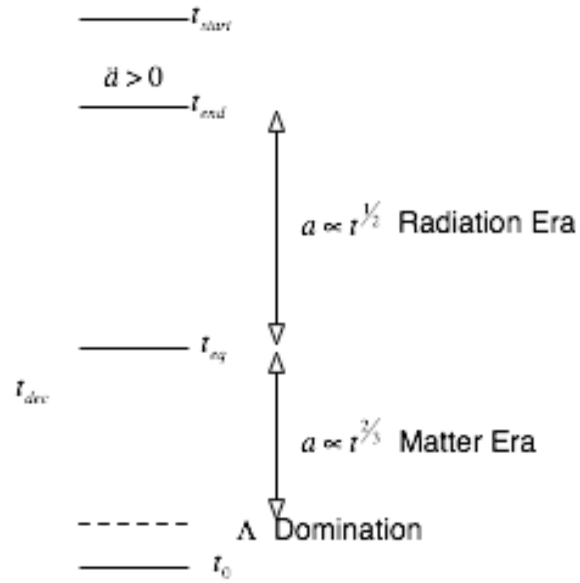
$$\frac{d_H(t_0)}{d_H(t_{rec})} \propto \frac{t_0}{t_{rec}} \approx \frac{3\times 10^{17} h^{-1}\,\text{sec}}{5\times 10^{12}\left(\Omega_m h^2\right)^{-1/2}\,\text{sec}} \approx 10^5$$

So the region which our current causal horizon comes from contains approximately 10^5 causally disconnected horizons at t_{rec}. However, the CMB has an almost uniform temperature over the current horizon, with $\frac{\Delta T}{T} \approx 10^{-5}$. This is called the Horizon Problem.

2. Inflation

2.1 Definition of Inflation

Inflation is a period of super-luminal expansion at very early times where $\ddot{a} > 0$ (NB: in a Λ universe, $\ddot{a} > 0$ although this happens at late times).



Since

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right),$$

then during inflation $P < -\frac{1}{3}\rho$. This means that the Strong Energy Condition is violated.

If $P = w\rho$, then $-1 \le w < -\frac{1}{3}$ → $1 + 3w < 0$.

As $\rho \propto a^{-3(1+w)}$, $H \propto a^{-\frac{3}{2}(1+w)}$ →

$$a \propto \begin{cases} t^{\frac{2}{3(1+w)}} & w \neq 0 \\ e^{Ht} & w = 0 \end{cases}$$

If $w = -1$, then we have de Sitter space.

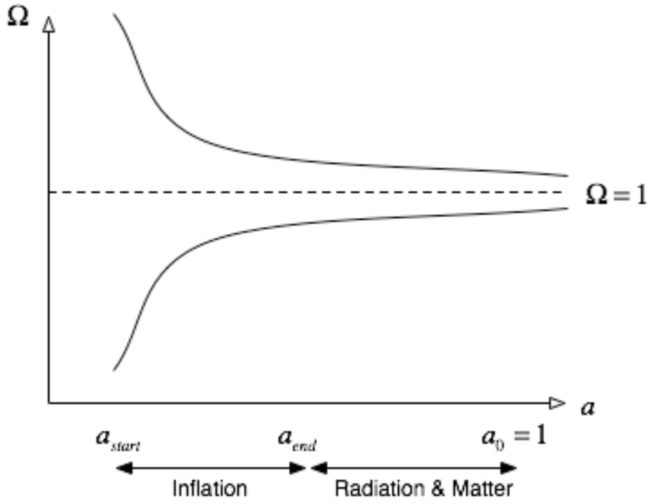
If $-1 < w < -\frac{1}{3}$, then we have power-law inflation, and if $w = -1$ we have exponential inflation.

One often quantifies the amount of inflation using the number of e-folds,

$$N = \log\left(\frac{a_{end}}{a_{start}}\right)$$

so e^N is the amount that a has increased during the inflation epoch.

Flatness Problem



Inflation is very small compared to radiation and matter.

In a radiation and matter dominated universe,

$$\dot{\Omega} = H\left(\frac{a + 2a_{eq}}{a + a_{eq}}\right)\Omega(\Omega - 1)$$

$$\rightarrow \int_{\Omega_{end}}^{\Omega_0} d\Omega\left(\frac{1}{(\Omega - 1)} - \frac{1}{\Omega}\right) = \int_{a_{end}}^{1} da\left(\frac{2}{a} - \frac{1}{a + a_{eq}}\right)$$

$$\rightarrow \frac{\Omega_0 - 1}{\Omega_0}\frac{\Omega_{end}}{\Omega_{end} - 1} = \frac{1}{a_{end}^{\ 2}}\frac{a_{end} + a_{eq}}{1 + a_{eq}} \approx \frac{a_{eq}}{a_{end}^{\ 2}} = \frac{T_{end}^{\ 2}}{T_\gamma(0)T_{eq}}$$

where the T 's are temperatures.

In an exponential inflation,

$$\dot{\Omega} = -2H\Omega(\Omega - 1)$$

$$\frac{\Omega_{end} - 1}{\Omega_{end}}\frac{\Omega_{start}}{\Omega_{start} - 1} = \left(\frac{a_{start}}{a_{end}}\right)^2$$

$$\rightarrow \frac{\Omega_0 - 1}{\Omega_0} = \frac{T_{end}^{\ 2}}{T_{eq}T_\gamma(0)}e^{-2N}\frac{\Omega_{start} - 1}{\Omega_{start}}$$

Assume $T_{end} \approx 10^{16} GeV$, $T_{eq} \approx 1eV$, $T_\gamma(0) \approx 10^{-4} eV$.

$$\frac{\Omega_0 - 1}{\Omega_0} = 10^{54} e^{-2N} \frac{\Omega_{start} - 1}{\Omega_{start}}$$

If N is sufficiently large $(N \approx 60)$, then $\Omega_0 \approx 1$ no matter what Ω_{start} was.

Horizon Problem

$$d_H(t) = a(t)\int_0^t \frac{dt'}{a(t')}$$

Change the variables to a, then;

$$d_H(a) = a\int_0^a \frac{da'}{|a'|H(a')}^2 \propto a\int_0^a da\, a^{\frac{1}{2}(3w-1)}$$

If $\frac{1}{2}(3w-1) \le -1$, i.e. $w \le -\frac{1}{3}$, then the integral becomes infinite, i.e.:

$$\text{Integral} = \begin{cases} a^{\frac{3}{2}(1+w)} & \propto H^{-1}, w > -\frac{1}{3} \\ \infty & w < -\frac{1}{3} \end{cases}$$

Particle horizon in a universe which is inflating is infinite. Hence, if the inflationary era is sufficiently long, then the entire observable universe can come from a single causally connected region.

Other aspects of inflation

1. $\rho_r \propto \frac{1}{a^4}$; $\rho_m \propto \frac{1}{a^3}$

   → matter and radiation are exponentially suppressed if $a \propto e^{Ht}$

2. $\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) = -\frac{\ddot{a}}{a} < 0$ during inflation

   Therefore during inflation the comoving Hubble radius is decreasing.

2.2 Potential Formulation & Slow Roll Dynamics

Consider a scalar field $\phi(\underline{x},t)$, which we will call the *Inflaton* (particle responsible for inflation). The action for this field is

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V|\phi|\right)$$

$$\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$$

i.e. the root of the determinant of the metric.

$$S = \int d^4x \underbrace{a^3}_{=\sqrt{-g}}\left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi)\right)$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\rho$$

where $\partial_\mu = \left(\frac{\partial}{\partial t}, \underline{\nabla}\right)$.

$$\rightarrow \; \rho\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\nabla\phi|^2 + V$$

$$P\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}|\nabla\phi|^2 - V$$

Compute the Euler-Lagrange Equations

$$\frac{\partial L}{\partial \dot{\phi}} = a^3\dot{\phi}$$

$$\frac{\partial L}{\partial(\nabla\phi)} = -a\nabla\phi$$

$$\frac{\partial L}{\partial \phi} = -a^3\frac{dV}{d\phi}$$

$$\frac{\partial}{\partial t}\left(a^3\dot{\phi}\right) - a\nabla^2\phi + a^3\frac{dV}{d\phi} = 0$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{dV}{d\phi} = 0$$

Switch to conformal time;

$$\phi'' + 3H'\phi' - \nabla^2\phi + a^2\frac{dV}{du}$$

where $H' = \frac{a'}{a}$, and the $'$ denote $\frac{\partial}{\partial\eta}$.

$$\rho\phi = \frac{1}{2a^2}\phi'^2 + \frac{1}{2}|\nabla\phi|^2 + V$$

$$P\phi = \frac{1}{2a^2}\phi'^2 - \frac{1}{6}|\nabla\phi|^2 - V$$