Gravitational Redshift
Consider a photon emitted with frequency ν_e by a distant source (i.e. $r \approx \infty$). Using 6.4, the frequency at r is given by:

$$\frac{\nu}{\nu_e} = \left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}}$$

As $r \to 2GM$, $\nu \to \infty$ and the photon is infinitely blue shifted.

Now consider a source hovering just above the event horizon at $r = 2GM + \varepsilon$, emitting a photon of frequency ν_e.

$$\frac{\nu}{\nu_e} = \left(\frac{1 - \frac{2GM}{2GM+\varepsilon}}{1 - \frac{2GM}{r}}\right)^{\frac{1}{2}} \approx \varepsilon^{\frac{1}{2}}\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} \to \varepsilon^{\frac{1}{2}} \, as \, r \to \infty$$

→ ν is reduced, i.e. red shifted, as it climbs out of the potential well.

7. FRW Universe

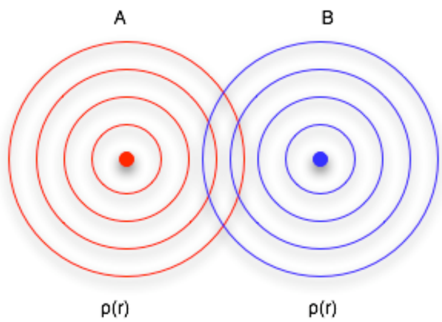
7.1 Expansion, Isotropy & Homogeneity

See Handout.

7.2 FRW Metric

One cannot derive the FRW metric directly from the observed expansion, isotropy and homogeneity.

Isotropy about two points → global isotropy and homogeneity.



r = distance from A.
s = distance from B.
Unfortunately, we do not have the luxury of having two independent observers. Hence we appeal to the *cosmological principle*, which is an extension of the Copernican Principle. It states that our place in space is typical.

The cosmological principle implies the existence of observers. Since we want to work within the framework of General Relativity, and hence that of the general principle of covariance, we need to be careful about these observers. We want a set of observers which are "commoving" which agree on the global cosmic time, t – this is often called Weyl's Postulate.

One could try to use static observers at the centre of galaxies, but galaxies have proper motions of order $\sim 100 km\, s^{-1}$ relative to the Hubble Flow (NB: the Hubble

Flow at $10Mpc$ is about $700 km\, s^{-1}$). Hence this would be an assumption. Conventionally, one now uses the CMBR to define this commoving rest frame.

NB: the existence of a preferred rest frame is required to define the notion of thermal equilibrium and have the Planck Spectrum of the CMB.

From these postulates (global homogeneity, isotropy and the existence of co-moving observers), is it possible to deduce that the metric that we want to look at is:

$$ds^2 = dt^2 - a^2(t)\, d\sigma_{III}{}^2$$

There are no cross-terms since they would introduce a preferred direction. $d\sigma_{III}{}^2$ is the metric on a 3-space which is homogeneous and isotropic. $a(t)$ is known as the scale factor.

There are three possibilities for $d\sigma_{III}{}^2$ which are compatible with the existence of 6 Killing vectors:

$(1,0,0);(0,1,0);(0,0,1)$ - translational symmetries

$(0,z,-y);(z,0,-x);(y,-x,0)$ - rotational symmetries.

The form of the metric most used in astronomy is:

$$d\sigma_{III}{}^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where k is a constant, typically scaled to $-1,0,1$ which represents curvature.

Surfaces of constant r are spheres of surface area $= 4\pi r^2$.

<u>$k = 0$</u>

If $k = 0$, then the Euclidean sections are:

$$d\sigma_{III}{}^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

This is flat Euclidean space.

<u>$k = -1$</u>

$$d\sigma_{III}{}^2 = \frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

If we set $r = \sinh\chi$, where $0 < \chi < \infty$

$$\rightarrow\ d\sigma_{III}{}^2 = d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Which is the metric on a hyperboloid.

If we say $\theta = const$, then we get:

→ negative curvature

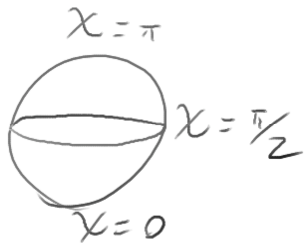
<u>$k=+1$</u>

$$d\sigma_{III}{}^2 = \frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

If we set $r = \sin\chi$, $0 \le \chi \le \pi$:

$$d\sigma_{III}{}^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Again set $\theta = const$.



→ Positive curvature
This is the metric on S^3.

Therefore,

$$ds^2 = dt^2 - a^2\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$
$$= dt^2 - a^2\left(d\chi^2 + S_k{}^2(\chi)\left[d\theta^2 + \sin^2\theta d\phi^2\right]\right)$$

where

$$S_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases}$$

NB: we can define $C_k(\chi) = \sqrt{1-kS_k{}^2(\chi)} = \begin{cases} \cosh\chi & k=-1 \\ 1 & k=0 \\ \cos\chi & k=+1 \end{cases}$

Now consider the introduction of a modified time coordinate η, defined by:

$$a d\eta = dt \rightarrow \eta = \int_0^t \frac{dt'}{a(t')}$$

η is called the conformal time.

$$ds^2 = a^2\left(d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

If $k=0$ with this time coordinate, the FRW universe is conformally Minkowski spacetime.

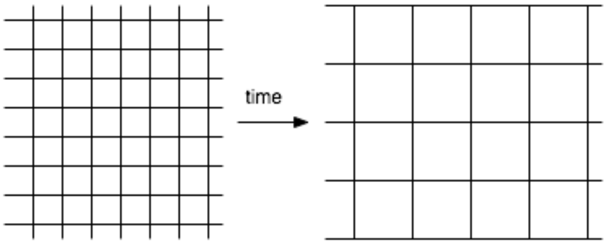
NB: one can make $d\sigma_{III}{}^2$ conformally Euclidean by setting

$$r = \frac{\tilde{r}}{1+\frac{k}{4}\tilde{r}}$$

$$\rightarrow\ d\sigma_{III}{}^2 = \frac{1}{\left(1+\frac{k}{4}\tilde{r}^2\right)^2}\left(d\tilde{r}^2 + \tilde{r}^2\left(d\theta^2 + \sin^2\theta^2 d\phi^2\right)\right)$$

To understand what is going on, suppress one of the dimensions.

$k = 0$:



<u>$k = 1$</u>

Consider drawing a rectangular grid on a balloon which you then inflate.

7.3 Friedman & Raychaudori Equations

One can show that:

$$R_{00} = -\frac{3\ddot{a}}{a}$$

$$R_{0i} = 0$$

$$R_{ij} = \left(2k + a\ddot{a} + 2\dot{a}^2\right)\gamma_{ij}$$

where γ_{ij} is the metric on the Euclidean sections, i.e.

$$\gamma_{ij} = diag\left(\frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right)$$

We will consider the effects of a homogeneous and isotropic perfect fluid.

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu}$$

$$U_\mu = (1,0)$$

$$T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = (\rho + P)U_\mu U_\nu - \frac{1}{2}g_{\mu\nu}(\rho - P)$$

<u>R_∞</u>:

$$-\frac{3\ddot{a}}{a} = 8\pi G\left(\rho + P - \frac{1}{2}(\rho - P)\right)$$

$$\rightarrow\ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$ - Raychauduri Equation

<u>R_{0i}</u>:

$$\left(2k + a\ddot{a} + 2\dot{a}^2\right)\gamma_{ij} = \frac{1}{2}a^2 8\pi G(\rho - P)\gamma_{ij}$$

$$\rightarrow\ \frac{2k}{a^2} + \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} = 4\pi G(\rho - P)$$

$$\rightarrow\ \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$ - Friedmann Equation.