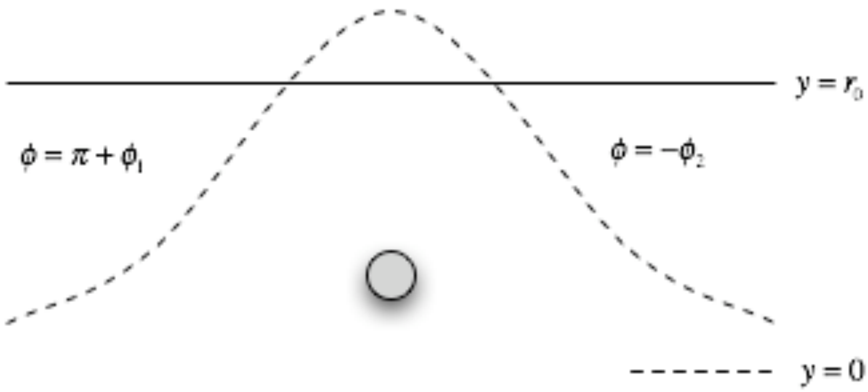
Last time: $u(\phi) = \sin\phi + \varepsilon\left(A\cos\phi + \frac{1}{2}\left(1+\cos^2\phi\right)\right) + O\left(\varepsilon^2\right)$.



At ∞, $r = \infty \rightarrow u = 0$. Hence we have,

$$0 = \sin\left(\pi + \phi_1\right) + \varepsilon\left(A\cos\left(\pi + \phi_1\right) + \frac{1}{2}\left(1 + \cos^2\left(\pi + \phi_1\right)\right)\right) + \ldots$$

$$0 = \sin\left(-\phi_2\right) + \varepsilon\left(A\cos\left(-\phi_2\right) + \frac{1}{2}\left(1 + \cos^2\left(-\phi_2\right)\right)\right) + \ldots$$

Assuming that ϕ_1 and ϕ_2 are small, then using small angle approximations we can say:

$$0 = -\phi_1 + \varepsilon\left(-A + \frac{1}{2}\times 2\right) + \ldots$$

$$0 = -\phi_2 + \varepsilon\left(A + \frac{1}{2}\times 2\right) + \ldots$$

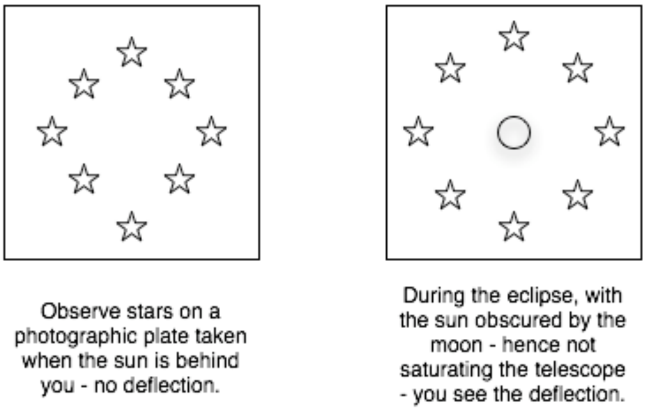
Adding these two together, the A's cancel and we have:

$$\delta = \phi_1 + \phi_2 = 2\varepsilon = \frac{2r_s}{r_0} = \frac{4GM}{r_0}$$

NB: in Newtonian spacetime, then $\phi_1 + \phi_2 = \frac{r_s}{r_0}$ i.e. half that predicted in the Schwarzschild spacetime.

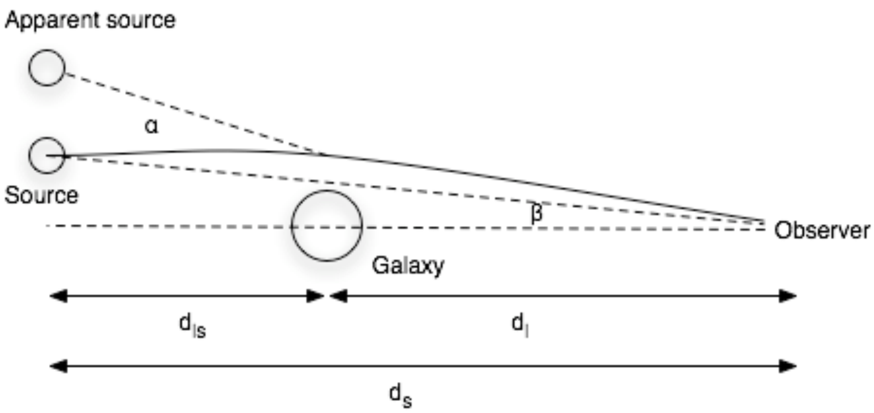
Einstein made this prediction which was tested in the famous 1919 Eclipse experiment organized by Eddington. The precise results of this experiment are now questioned, but it was thought to confirm Einstein's predictions at the time.

How did it work?



The size of deflection measured is about 1.5 arcseconds.

Gravitational Lensing



If $\beta = 0$, then axial symmetry exists and an Einstein ring is formed.

6.6 Perihelion Precession

If $L = 1$ (i.e. Particles) then $V(r) = \left(1 + \frac{h^2}{r^2}\right)\left(1 - \frac{r_s}{r}\right)$

→ $\left(\frac{dy}{d\phi}\right)^2 = \frac{E^2 - 1}{h^2} - u^2 + \frac{r_s}{h^2}u + r_s u^3$ where $u = \frac{1}{r}$.

In Newtonian Gravity, $\left(\frac{dy}{d\phi}\right)^2 = \frac{E^2 - 1}{h^2} - u^2 + \frac{r_s}{h^2}u$. We want to work out the perturbations which are due to the extra term in the equation – i.e. $r_s u^3$.

Define $u = v + \hat{v}$ and use $\hat{v}$ to eliminate the linear term in v.

Staying with the Newtonian limit:

$$\left(\frac{dv}{d\phi}\right)^2 = \left(\frac{E^2 - 1}{h^2} - \hat{v}^2 + \frac{r_s}{h^2}\hat{v}\right) + \left(\frac{r_s}{h^2} - 2\hat{v}\right)v - v^2$$

If we define $\hat{v} = \frac{r_s}{2h^2}$ then

$$\left(\frac{dv}{d\phi}\right)^2 = \left(\frac{E^2 - 1}{h^2} + \frac{r_s^{\,2}}{4h^4}\right) - v^2$$

$$\rightarrow \; v = \frac{r_s}{2h^2}\left(1 + \frac{4\left(E^2 - 1\right)h^2}{r_s^{\,2}}\right)^{1/2} \cos\left(\phi - \phi_0\right)$$

We can write this in the form

$$\frac{\ell}{r} = 1 + e\cos\left(\phi - \phi_0\right)$$

where $\ell = \frac{2h^2}{r_s}$ and $e = \left(1 + \frac{4\left(E^2 - 1\right)h^2}{r_s^{\,2}}\right)^{1/2}$

Choose for convenience $\phi_0 = 0$.

$$\begin{aligned} \ell &= r + er\cos\phi \\ &= \sqrt{x^2 + y^2} + ex \end{aligned}$$

for $x = r\cos\phi$ and $y = r\sin\phi$.

$$\ell^2 = \left(1 - e^2\right)x^2 + 2\ell ex + y^2$$

This is an ellipse if $0 < e^2 < 1$, otherwise it is a conic section. $\varepsilon = 0$ is a circle, $\varepsilon = 1$ is a parabola, and $\varepsilon > 1$ is a hyperbola.

Now do the same for general relativity.

$$\left(\frac{dv}{d\phi}\right)^2 = \left(\frac{E^2 - 1}{h^2} + \frac{r_s}{h^2}\hat{v} - \hat{v}^2 + r_s\hat{v}^3\right) + \left(\frac{r_s}{h^2} - 2\hat{v} + 3r_s\hat{v}^2\right)v - \left(1 - 3r_s\hat{v}\right)v^2 + r_s v^3$$

Choose $\hat{v}$ such that $3r_s\hat{v}^2 - 2\hat{v} + \frac{r_s}{h^2} = 0$. Hence

$$\hat{v} = \frac{1 \pm \left(1 - \frac{3r_s^{\,2}}{h^2}\right)^{1/2}}{3r_s} \approx \frac{r_s}{2h^2}\left(1 + \frac{9r_s^{\,2}}{4h^2} + \ldots\right),$$

taking the negative root. Now set

$$\alpha^2 = \frac{E^2 - 1}{h^2} - \hat{v} + \frac{r_s}{h^2}\hat{v} + r_s\hat{v}^3$$

and

$$\omega^2 = 1 - 3r_s\hat{v} \approx 1 - \frac{3r_s^{\,2}}{2h} + O\left(\left(\frac{r_s}{h}\right)^4\right)$$

$$\rightarrow \; \omega \approx 1 - \frac{3r_s^{\,2}}{4h^2}$$

then

$$\left(\frac{dv}{d\phi}\right)^2 = \alpha^2 - \omega^2 v^2 + r_s v^3$$

To lowest order, the solution is given by

$$v = \alpha\cos\left(\omega\left(\phi-\phi_0\right)\right)$$

$$\rightarrow \frac{\ell}{r} = 1 + e\cos\left(\omega\left(\phi-\phi_0\right)\right)$$

where $\ell = \frac{1}{\hat{v}} \approx \frac{2h^2}{r_s} + O(1)$ and $\varepsilon = \frac{\alpha}{\hat{v}} \approx \left(1+\frac{4\left(E^2-1\right)h^2}{r_s^{\,2}}\right)^{1/2} + O\left(\frac{r_s}{h^2}\right)$. This is an ellipse with an angular shift of the orbit

$$\Delta\phi = \frac{2\pi}{\omega} - 2\pi$$

$$\approx 6\pi\left(\frac{GM}{h}\right)^2$$

This is called Perihelion Precession.

For Mercury, $\Delta\phi = 43''$ (43 arcseconds) per Earth century. It has been observed to be $41.5''$.
Note that this doesn't happen in Newtonian gravity; just GR. The error is due to the fact that the sun is not a perfect sphere.

6.7 Black Holes

Newtonian Argument

Consider a particle of mass m on the surface of an object of mass M and radius R. At position $r > R$, the energy of the particle is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

The escape velocity is defined to be the minimum velocity required to get from $r = R$ to $r = \infty$. That is the velocity required to get $E = 0$ at $r = \infty$.

$$\rightarrow v^2 = \frac{2GM}{R} = \frac{r_s}{R}$$

If $v^2 = 1$ (i.e. the speed of light), then $R = 2GM = r_s$. Hence, an object with radius r_s cannot be escaped by a particle moving less than the speed of light.

NB: this argument applies to a particle of mass m, not to a photon. We need relativity to deal with $v \sim c$ and to model photons.

Singularities

The metric for the two-sphere $\left(S^2\right)$ has a singularity.

$$g_{\mu\nu} = diag\left(1,\sin^2\theta\right)$$

$$g^{\mu\nu} = diag\left(1,\cos ec^2\theta\right)$$

and $\cos ec \to \infty$ at θ, π. In this case, one can regularize the singularity by a choice of coordinates.
The Schwarzschild metric has a singularity at $r = 2GM$, where $g_{rr} \to \infty$ and $g^{tt} \to \infty$, which can be removed by a coordinate transformation. Such singularities are called "removable".

A "true" singularity is defined by the fact that curvature invariance such as R, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ are infinite. For example, in the Schwarzschild metric,

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48(GM)^2}{r^6}$$

which tends to infinity at $r = 0$. Hence $r = 0$ is a true singularity (but $r = 2GM$ is not).

<u>Timelike ↔ Spacelike</u>

Consider the $t - r$ part of the Schwarzschild metric.

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

For $r > 2GM$, $g_{tt} > 0$ → t is a timelike coordinate, and $g_{rr} < 0$ is a spacelike coordinate. Whereas for $r < 2GM$, $g_{tt} < 0$ and $g_{rr} > 0$ → t is now spacelike, and r is now timelike.

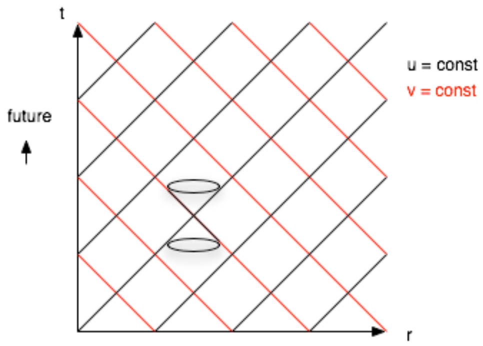
<u>Eddington-Finkelstein Coordinates</u>

In Minkowski space we can define null coordinates $u = t - r$, $v = t + r$ so that

$$ds^2 = dudv - \frac{1}{4}(v - u)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Along a radially ingoing null geodesic v is constant, and on outgoing null geodesics u is constant.

Suppress θ and ϕ directions, so that each point represents a 2-sphere.



$u = const$ and $v = const$ meet at $90°$ and define the future-pointing light cones.

We want to do something similar in the Schwarzschild spacetime.

Consider a radially infalling null geodesic.

$$ds^2 = \dot{\theta} = \dot{\phi} = 0 \rightarrow \dot{r}^2 = E^2 = \dot{t}^2\left(1 - \frac{2GM}{r}\right)^2$$

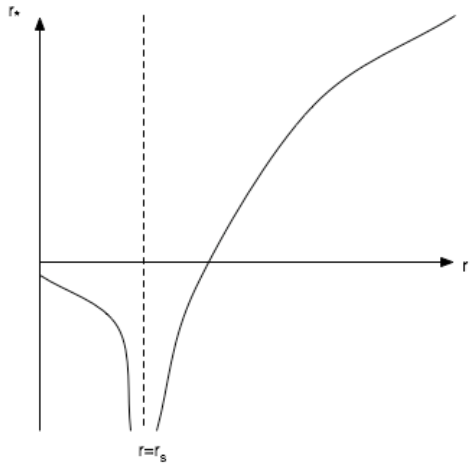
$$\rightarrow \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{r}\right)^2$$

$$\rightarrow t = \pm\left(r + 2GM\log|r - 2GM|\right)$$

The $\pm$ correspond to $\frac{dr}{dt} > 0$ and $\frac{dr}{dt} < 0$ (incoming and outgoing directions respectively).

Define r_* such that $dr_* = \left(1 - \frac{2GM}{r}\right)^{-1} dr$, then

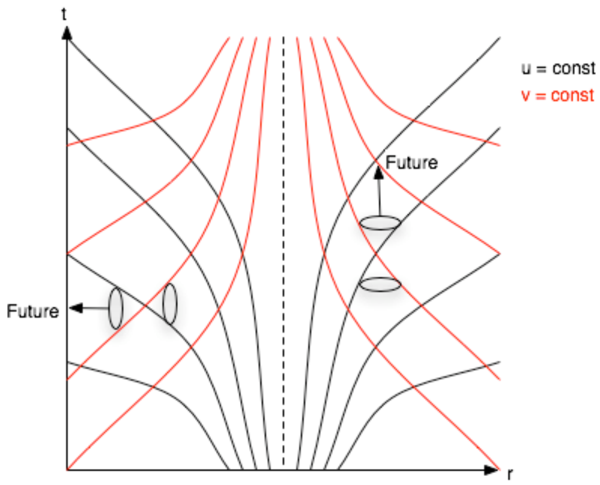
$$r_* = r + 2GM \log\left(\frac{r}{2GM} - 1\right)$$



The Eddington-Finkelstein coordinates are given by $u = t - r_*$ and $v = t + r_*$, which represent the outgoing and incoming null geodesics in analogy with Minkowski spacetime.
The $t - r$ part of the metric is

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$
$$= \left(1 - \frac{2GM}{r}\right)dudv$$

which is non-singular at $r = 2GM$.



The lines $u = const$ and $v = const$ still meet at $90°$ (the $t - r$ part of the metric is conformally Minkowski space), but the future light cones do not point in the positive t-direction for $r < 2GM$. They are rotated by $90°$ as they go through $r = 2GM$. Any particle which crosses from $r > 2GM$ to $r < 2GM$ cannot return. Even the outgoing light rays $u = const$ are "ingoing", and everything reaches $r = 0$ in a finite proper time.
$r = 2GM$ is known as the Event Horizon.

Near $r = 0$, tidal forces are extremely large. $R_{\mu\nu\alpha\beta} \sim \frac{1}{r^3}$, and any material will be destroyed.