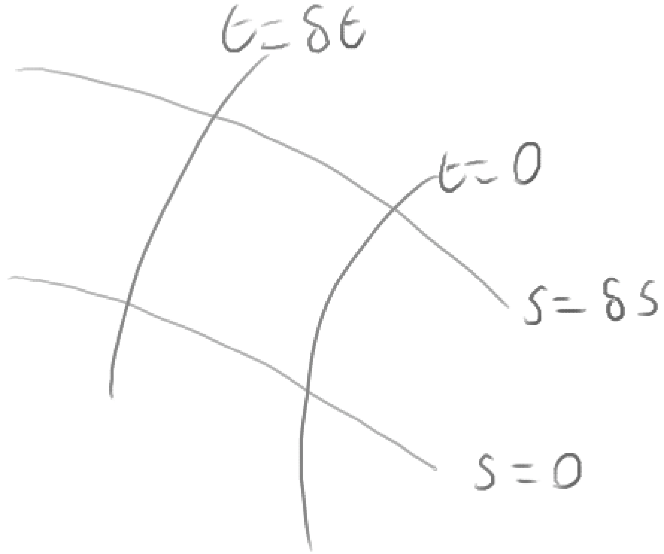
PC4771 – Gravitation – Lectures 13 & 14

4.4 Geodesic Deviation



$$T^\mu = \frac{dx^\mu}{dt}$$

$$S^\mu = \frac{dx^\mu}{ds}$$

There are four possible second derivatives:

$$\frac{d^2x^\mu}{dt^2} = T^\alpha \partial_\alpha T^\mu = \frac{dx^\alpha}{dt}\frac{\partial}{\partial x^\alpha}\left(\frac{dT^\mu}{dt}\right)$$

$$\frac{d^2x^\mu}{ds^2} = S^\alpha \partial_\alpha S^\mu$$

$$\frac{d^2x^\mu}{dsdt} = S^\alpha \partial_\alpha T^\mu$$

$$\frac{d^2x^\mu}{dtds} = T^\alpha \partial_\alpha S^\mu$$

Of course, $\frac{d^2x^\mu}{dsdt} = \frac{d^2x^\mu}{dtds}$ as the derivatives wrt s and t commute.

→ $S^\alpha \partial_\alpha T^\mu = T^\alpha \partial_\alpha S^\mu$

→ $S^\alpha \nabla_\alpha T^\mu = T^\alpha \nabla_\alpha S^\mu$ if $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$.

This means that it doesn't matter which way around the loop you go.

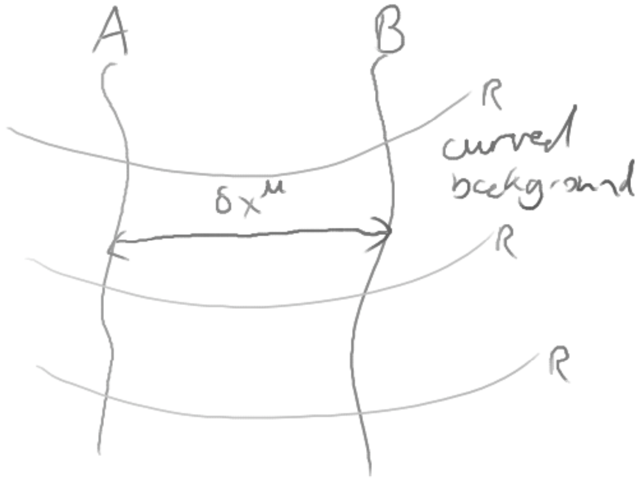
Now consider

$$\begin{aligned}
T^\alpha \nabla_\alpha \left(T^\beta \nabla_\beta S^\mu\right) &= T^\alpha \nabla_\alpha \left(S^\beta \nabla_\beta T^\mu\right) \\
&= \left(T^\alpha \nabla_\alpha S^\beta\right)\left(\nabla_\beta T^\mu\right) + T^\alpha S^\beta \nabla_\alpha \nabla_\beta T^\mu \\
&= \left(S^\alpha \nabla_\alpha T^\beta\right)\left(\nabla_\beta T^\mu\right) + T^\alpha S^\beta \left[\nabla_\beta \nabla_\alpha T^\mu + R^\mu{}_{\rho\alpha\beta} T^\rho\right] \\
&= S^\beta \nabla_\beta \underbrace{\left(T^\alpha \nabla_\alpha T^\mu\right)}_{=0} + R^\mu{}_{\rho\alpha\beta} T^\rho T^\alpha S^\beta
\end{aligned}$$

$\left(T^{\alpha}\nabla_{\alpha}T^{\mu}\right)$ is zero for an affinely parameterized geodesic.
→ $\left(T^{\alpha}\nabla_{\alpha}\right)\left(T^{\beta}\nabla_{\beta}\right)S^{\mu}=R^{\mu}{}_{\rho\alpha\beta}T^{\rho}T^{\alpha}S^{\beta}$ along an affinely parameterized geodesic.

Consider two particles A and B moving along geodesics parameterized by t. If δx^{μ} is the difference between them i.e. $S^{\mu}\left(=\frac{dx^{\mu}}{ds}\delta s\right)=\delta x^{\mu}$, then

$$\frac{d^2}{d\tau^2}\delta x^{\mu}=R^{\mu}{}_{\rho\alpha\beta}T^{\rho}T^{\alpha}T^{\beta}\delta x^{\beta}.$$



This is the equation of geodesic deviation.

5. Einstein Equations

5.1 Energy-Momentum Tensor

Consider the tensor $T^{\mu\nu}=(\rho+P)U^{\mu}U^{\nu}-P\eta^{\mu\nu}$, and assume that it is conserved, where:s

- ρ is density
- P is pressure
- $U^{\mu}=\gamma(1,\underline{v})$ of special relativity

$$\partial_{\mu}T^{\mu\nu}=\partial_{\mu}\left[(\rho+P)U^{\mu}\right]\cdot U^{\nu}+(\rho+P)U^{\mu}\partial_{\mu}U^{\nu}-\partial^{\nu}P$$

In the non-relativistic limit:

1. $U^{\mu}=\left(1,v^{i}\right)+O\left(v^2\right)$
2. $\rho>>P$
3. $v\dot{P}<<|\nabla P|$, so pressure gradients dominate.

Consider $\nu=0$ and $\nu=i$ as separate cases.

$\nu=0$:

$\partial_{\mu}\left[(\rho+P)\right]U^{\mu}-\dot{P}=0$

→ $\dot{\rho}+\rho\underline{\nabla}\cdot\underline{v}=0$ (1)

This is the energy conservation / continuity equation.

$\nu = i$:

$$\left(\partial_\mu (\rho + P) U^\mu\right) v^i + (\rho + P)\left(\dot{v}^i \partial_j v^i\right) - \partial^j P = 0$$

Substitute the continuity equation into this, and the first term drops out, leaving

$$\rho\left[\dot{v}^i + v^j \partial_j v^i\right] = \partial^j P$$

$$\rightarrow \rho\left(\frac{d\underline{v}}{dt} + (\underline{v} \cdot \underline{\nabla})\underline{v}\right) = -\underline{\nabla} P \quad (2)$$

This is the momentum conservation, aka Newton's second law.

These two equations are the equations of fluid dynamics.

$T^{\mu\nu}$ is the energy-momentum tensor for a perfect fluid. In general,

$$T^{\mu\nu} = \left(\begin{array}{c|c} T^{00} & T^{i0} \\ \hline T^{0i} & T^{ij} \end{array}\right)$$

T^{00} is the energy gdensity.
T^{i0} and T^{0i} is the momentum flux.
Down the diagonal part of T^{ij} is the pressure.
Off the diagonal of T^{ij} is the stress, or anisotropic pressure.
Another example is electromagnetism.

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

(See 1.4.6)

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\beta} F^{\rho}{}_{\nu} + \frac{1}{2} g_{\mu\nu} F^2\right)$$

where $F^2 = F_{\alpha\beta} F_{\alpha\beta}$.

$T_{00} = \frac{1}{8\pi}\left(\underline{E}^2 + \underline{B}^2\right)$ - energy density

$T_{0i} = \frac{1}{4\pi}\left(\underline{E} \times \underline{B}\right)$ - Poynting Vector (momentum flux of the electromagnetic field)

If we take $\partial_\mu T^{\mu\nu} = 0$, then we get the Maxwell equations in ElectroDynamics.

In special relativity, energy and momentum conservation are given by $\partial^\mu T_{\mu\nu} = 0$.
In GR, $\nabla^\mu T_{\mu\nu} = 0$ and $T^{\mu\nu} = (\rho + P) U^\mu U^\nu - P g^{\mu\nu}$ for a perfect fluid.

The energy momentum tensor will play the role of a generalized concept of mass.
For a perfect fluid, the Weak Energy Condition implies that $\rho > 0$ and $\rho + P > 0$.
Also, the Strong Energy Condition implies that $\rho + 3P > 0$.

5.2 SEP (Strong Equivalence Principle), Einstein Tensor & Einstein Equations

Newtonian gravity
$\phi \equiv$ gravitational potential ("if you know this, you know everything")
$\underline{F} = -\underline{\nabla}\phi \equiv$ gravitational force
$\underline{\nabla}^2\phi = 4\pi G\rho \equiv$ Poisson equation – links ϕ to ρ. (c.f. electomagnetism).
Gravitational force can be set to zero at some point → existence of freely falling frames. But tidal gravitational forces $\frac{\partial F_i}{\partial x_j} = \frac{d^2\phi}{dx_i dx_j}$ cannot be removed (see examples 1, question 7). [Example of tidal gravitational forces: the effect of the moon on the ocean, i.e. tides]

Recall SEP
Coordinates in a lift are x^μ, coordinates relative to the ground are x'^μ.
SEP → the observer in the lift experiences special relativity - $\frac{d^2x^\mu}{d\tau^2} = 0$ (2) and

$$\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \begin{cases} 1 & particles \\ 0 & light \end{cases} \quad (1)$$

Consider the coordinate transformation.

$$dx^\mu = J^\mu{}_\nu dx'^\nu$$

$$\rightarrow \frac{dx^\mu}{d\tau} = J^\mu{}_\nu \frac{dx'^\nu}{d\tau}$$

Hence equation 1 becomes $n_{\alpha\beta}J^\alpha{}_\mu J^\beta{}_\nu \frac{dx'^\mu}{d\tau}\frac{dx'^\nu}{d\tau} = 1 \; or \; 0$

If we define $g_{\mu\nu} = J^\alpha{}_\mu J^\beta{}_\nu \eta_{\alpha\beta}$ then:

$$g_{\mu\nu}\frac{dx'^\mu}{d\tau}\frac{dx'^\nu}{d\tau} = 1 \; or \; 0$$

Hence equation 2 becomes:

$$\frac{d^2x^\mu}{d\tau^2} = J^\mu{}_\nu \frac{d^2x'^\nu}{d\tau^2} + \partial'_\gamma J^\mu{}_\nu \frac{dx'^\gamma}{d\tau}\frac{dx'^\nu}{d\tau} = 0$$

$$\rightarrow J^\mu{}_\nu \frac{d^2x'^\nu}{d\tau^2} + J^\rho{}_\gamma \partial_\rho J^\mu{}_\nu \frac{dx'^\gamma}{d\tau}\frac{dx'^\nu}{d\tau} = 0$$

$$\rightarrow \frac{d^2x'^\alpha}{d\tau^2} + \left(J'^\alpha{}_\mu J^\rho{}_\gamma \partial_\rho J^\mu{}_\nu\right)\frac{dx'^\gamma}{d\tau}\frac{dx'^\nu}{d\tau} = 0$$

Note that $\Gamma'^\alpha{}_{\gamma\nu} = J'^\alpha{}_\mu J^\rho{}_\gamma J^\beta{}_\nu \underbrace{\Gamma^\mu{}_{\rho\beta}}_{=0} + J'^\alpha{}_\mu J^\rho{}_\gamma \partial_\rho J^\mu{}_\nu = J'^\alpha{}_\mu J^\rho{}_\gamma \partial_\rho J^\mu{}_\nu$

Hence $\frac{d^2x'^\alpha}{d\tau^2} + \Gamma'^\alpha{}_{\gamma\nu}\frac{dx'^\gamma}{d\tau}\frac{dx'^\nu}{d\tau} = 0$
From the point of view of an observer on the ground, the equation of motion is the geodesic equation.

→ our new theory of gravity will replace ϕ with $g_{\mu\nu}$ and $\underline{F} = -\underline{\nabla}\phi$ with $\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ → test particles and light rays will move on geodesics.
And the Poisson equation will be replaced by the Einstein Equation.

Note: we have already shown that in a LIF $\Gamma^{\mu}{}_{\alpha\beta}$ can be set to zero at a point, but that $\partial_{\rho}\Gamma^{\mu}{}_{\alpha\beta}$ cannot. This is equivalent to removing the gravitational force at that point, but not the tidal fields. It hints that tidal fields are encoded in the derivative of $\partial_{\rho}\Gamma^{\mu}{}_{\alpha\beta} \sim R^{\mu}{}_{\rho\alpha\beta}$.

One cannot derive the Einstein Equation. It is a particular choice which works in practice. It fulfils various simple criteria.

1. It relates the metric, plus first and second derivatives, to the energy-momentum tensor.
   → $G\left(g_{\mu\nu}, \partial g_{\mu\nu}, \partial^2 g_{\mu\nu}\right) \simeq T$
2. It must be coordinate-independent.
   → it must be tensorial.
3. Energy-Momentum has to be conserved.
   → $\nabla_{\mu} T^{\mu\nu} = 0$
4. It should become the Poisson equation in the non-relativistic limit. (see 5.3)

(1) and (2) imply that $G = G\left(g_{\mu\nu}, R_{\mu\nu}, R, R_{\mu\nu\rho\sigma}\right)$

(3) → if $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where G is the gravitational constant, then $\nabla^{\mu} G_{\mu\nu} = 0$.

Therefore $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is a suitable choice.

→ Einstein Equation: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$