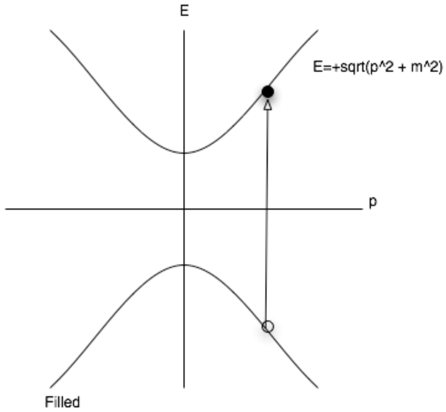
Solutions $E > m$, $\underline{p}$, $s\left(=+\frac{1}{2},-\frac{1}{2}\right)$
Plus solution $-E < -m$, $-\underline{p}$, $-s$.

3.4 Dirac hole theory

Spin-1/2 particles are fermions, and they obey the exclusion principle.
Imagine that the vacuum is a completely filled 'sea' of negative energy states.



Positive energy e^- can't fall down to negative energy states because of the exclusion principle.

But you can excite a particle from the sea. The hole looks like an antiparticle, i.e. absence of e^- with $-E,-\underline{p},-s$ looks like e^+ with $E,\underline{p},s$. This is a way to live with negative energy states – for a while!

3.5 Relativistic Covarience

3.5.1 Covariant Notation

Consider the Dirac equation,

$$i\frac{\partial\psi}{\partial t} = \left[-i\underline{\alpha}\cdot\underline{\nabla} + \beta m\right]\psi \quad (1)$$

This equation is not obviously covariant.

Multiply by β from the left, and define $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$ ($i = 1,2,3$).

$$i\gamma^0\frac{\partial\psi}{\partial x^0} = \left[-i\gamma^i\frac{\partial}{\partial x^i} + m\right]\psi$$

since $\beta^2 = 1$. Writing $\gamma^\mu = \left(\gamma^0,\gamma^i\right)$, then (1) becomes

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0 \quad (2)$$

where the γ-matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad (3)$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad (4a)$$

$$\text{or } \gamma^{0\dagger} = \gamma^0,\ \gamma^{i\dagger} = -\gamma^i \quad (4b)$$

which follow from the properties of β, α_i. e.g. $\left(\gamma^i\right)^\dagger = \left(\beta\alpha_i\right)^\dagger = \alpha_i\beta = -\beta\alpha_i = -\gamma^i$.

It's also useful to introduce the (Dirac) adjoint

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0 \quad (5)$$

The charge current density $j^\mu(x) = \left(\rho, \underline{j}\right)$ can be rewritten (exercise)

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x) \quad (6)$$

with the equation of continuity

$$\partial_\mu j^\mu(x) = 0 \quad (7).$$

This all looks covariant – but is it?
If they are valid physical equations, they must be by the principle of special relativity.
But what are the transformation properties of ψ ?

3.5.2 Proof of covariance

Consider a Lorentz transformation from $O \to O'$ such that $x \to x' = ax$ $\left(\Lambda_\mu{}^\nu \to a_\mu{}^\nu\right)$
i.e. $x'^\mu = a^\mu{}_\nu x^\nu$ (1) where $a^\mu{}_\nu a_\mu{}^\sigma = g_\nu{}^\sigma = \delta_\nu{}^\sigma$ (2).

What is the transformation law which takes $\psi(x) \to \psi'(x')$?

Define $S(a)$ by

$$\psi'(x') = S(a)\psi(x) \quad (3)$$

and correspondingly,

$$\psi(x) = S^{-1}(a)\psi'(x') = S\left(a^{-1}\right)\psi'(x') \quad (4)$$

Then the Dirac equation is covariant and $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ is a 4-vector, provided:

$$S^{-1}(a)\gamma^\mu S(a) = a^\mu{}_\nu\gamma^\nu \quad (5)$$

and

$$S^{-1}(a) = \gamma^0 S(a)\gamma^0 \quad (6).$$

To prove this is so, we need to show that the Dirac equation is invariant in form under Lorentz transformation plus (5) and (6), i.e.

$$\left(i\gamma^\mu\partial_\mu' - m\right)\psi'(x') = 0 \quad (7)$$

implies

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0$$

(or vice versa) where $\partial'_\mu = \dfrac{\partial}{\partial x^{\mu\,\prime}} = a_\mu{}^\sigma\partial_\sigma$. (8)

Substitute (3) and (8) into (7).

$$\rightarrow \left(i\gamma^\mu a_\mu{}^\sigma\partial_\sigma - m\right)S(a)\psi(x) = 0$$

Multiply this from the left by $S^{-1}(a)$.

$$\rightarrow \left(iS^{-1}(a)\gamma^\mu S(a)a_\mu{}^\sigma\partial_\sigma - \underbrace{\left(S^{-1}(a)S(a)\right)}_{=I}m\right)\psi(x) = 0$$

But by (5), $S^{-1}(a)\gamma^{\mu}S(a)a_{\mu}{}^{\sigma} = \gamma^{\nu}\underbrace{a^{\mu}{}_{\nu}a_{\mu}{}^{\sigma}}_{\delta_{\nu}{}^{\sigma}} = \gamma^{\sigma}$ by (2).

So (9) becomes

$$\left(i\gamma^{\sigma}\partial_{\sigma} - m\right)\psi(x) = 0$$

as required.

What about $j^{\mu}(x) = \overline{\psi}\gamma^{\mu}\psi(x)$?
To show it's a 4-vector, we need to show

$$\overline{\psi}'(x')\gamma^{\mu}\psi'(x') = a^{\mu}{}_{\nu}\overline{\psi}(x)\gamma^{\nu}\psi(x).$$

Consider the transformation law of $\overline{\psi}'(x')$.
$\psi'(x') = S(a)\psi(x) \rightarrow \psi'^{\dagger}(x') = \psi^{\dagger}(x)S^{\dagger}(a)$
Therefore $\overline{\psi}'(x') = \psi^{\dagger}{}'(x')\gamma^{0} = \psi^{\dagger}(x)\gamma^{0}\gamma^{0}S^{\dagger}(a)\gamma^{0} = \overline{\psi}(x)\gamma^{0}S^{\dagger}(a)\gamma^{0}$
So $\overline{\psi}'(x') = \overline{\psi}(x)S^{-1}(a)$ by (6) (10)
Therefore,

$$\begin{aligned}\overline{\psi}'(x')\gamma^{\mu}\psi'(x') &= \overline{\psi}(x)S^{-1}(a)\gamma^{\mu}S(a)\psi(x)\\ &= a^{\mu}{}_{\nu}\overline{\psi}(x)\gamma^{\nu}\psi(x)\end{aligned}$$

which is equation (9). QED.
Note that (10) also implies $\overline{\psi}(x)\psi(x)$ is a Lorentz scalar, etc.

3.5.3 Lorentz boost

Consider $O \rightarrow O'$
$(x,y) \rightarrow (x',y')$ with difference vt. Then $x' = ax$ where

$$a = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0\\ \sinh\omega & \cosh\omega & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

where $\sinh\omega = \frac{v}{\sqrt{1-v^2}}$ and $\cosh\omega = \frac{1}{\sqrt{1-v^2}}$. Then

$S(a) = e^{\omega a_1/2} = I\cosh\frac{\omega}{2} + \alpha_1 \sinh\frac{\omega}{2}$ where we have used the fact that $\alpha_1{}^2 = 1$; $\alpha_1{}^{2n} = 1$; $a_1{}^{2n+1} = \alpha_1$.

This satisfies the conditions (5) and (6). (Exercise – verify this).

Note – for particles at rest, $\psi \propto \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ etc., and we can use a Lorentz boost to get $\psi(v)$.