3.2 Angular Momentum & Spin

Compare with orbital angular momentum.

$$\underline{L} = \underline{x} \times \underline{p}$$

or

$$L_i = \varepsilon_{ijk} x_j p_k$$

with j,k summed over.

$$\varepsilon_{ijk} = \begin{cases} 1 & cyclic\ permutation\ of\ 1,2,3 \\ -1 & non-cyclic\ permutation\ of\ 1,2,3 \\ 0 & if\ any\ two\ are\ equal \end{cases}$$

Is orbital angular momentum conserved? If so, need $\left[H,\underline{L}\right]=0$ where $H = -i\underline{\alpha}\cdot\underline{\nabla} + \beta m = \underline{\alpha}\cdot\underline{p} + \beta m$ and $\left[x_i, p_j\right] = i\delta_{ij}$.

Compare this with

$$\begin{aligned}\left[H,L_j\right] &= \left[\alpha_i p_i, \varepsilon_{jkl} x_k p_l\right] \\ &= \varepsilon_{jkl}\alpha_i\left[p_i, x_k\right]p_l \\ &= \varepsilon_{jkl}\alpha_i\left(-i\delta_{ik}\right)p_l \\ &-i\varepsilon_{jkl}\alpha_k p_l\end{aligned}$$

Therefore, $\left[H,\underline{L}\right] = -i\underline{\alpha}\times\underline{p}$. $\underline{L}$ is not conserved. But total angular momentum must be conserved! Cf. the matrix operators,

$$\Sigma_j = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_j \end{pmatrix}$$

Can show (exercise),

$$\left[\alpha_i, \Sigma_j\right] = 2i\varepsilon_{ijk}\alpha_k$$

and

$$\left[\beta, \Sigma_j\right] = 0$$

So,

$$\left[H,\Sigma_j\right] = \left[\alpha_i p_i + \beta m, \Sigma_j\right] = 2i\varepsilon_{ijk}\alpha_k p_i = 2i\varepsilon_{ijk}\alpha_k p_i$$

i.e. $\left[H,\underline{\Sigma}\right] = 2i\underline{\alpha}\times\underline{p}$. Hence,

$$\left[H, \underline{L} + \frac{1}{2}\underline{\Sigma}\right] = 0$$

i.e. $\underline{J} = \underline{L} + \underline{S}$ is conserved, where the spin operator $\underline{S} = \frac{1}{2}\underline{\Sigma}$. Further, since $\Sigma_i^{\,2} = 1$, the eigenvalues of S_i are $\pm 1/2$ → particle has spin-1/2.

3.3 Plane Wave States

Look for solutions

$$\psi(x) = Ne^{-ipx}U\left(\underline{p}\right)$$

where $U\left(\underline{p}\right)$ is a 4-component spinor to be determined, and $p^2 = E^2 - m^2$ to satisfy the KE equation.

Use Dirac representation.

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } U = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where ξ, η are 2-component objects. Substitute into the Dirac equation.

$$i\frac{\partial \psi}{\partial t} = -i\underline{\alpha} \cdot \underline{\nabla}\psi + \beta m \psi$$

and use

$$\partial_\mu e^{-ipx} = \left(\frac{\partial}{\partial t}, \underline{\nabla}\right) e^{-i(Et - \underline{p}\cdot\underline{x})}$$
$$= -i\left(E, -\underline{p}\right) e^{-ipx}$$

$$\rightarrow\ E\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} m & \underline{\sigma}\cdot\underline{p} \\ \underline{\sigma}\cdot\underline{p} & -m \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

or

$$E\xi = m\xi + \underline{\sigma}\cdot\underline{p}\eta \quad (1)$$
$$E\eta = \underline{\sigma}\cdot\underline{p}\xi - m\eta \quad (2)$$
$$\rightarrow\ \eta = \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\xi$$

Substitute this into (1);

$$(E+m)(E-m)\xi = \left(\underline{\sigma}\cdot\underline{p}\right)^2 \xi$$
$$\rightarrow\ \left(E^2 - m^2\right)\xi = \underline{p}^2 \xi \quad (3)$$

cf. positive energy $E\left(\underline{p}\right) = +\sqrt{m^2 + \underline{p}^2}$ (i.e. $\left|\sqrt{m^2 + \underline{p}^2}\right|$).

There are two solutions to (3):

$$\xi_s = \sqrt{E\left(\underline{p}\right) + m}\,\chi_s$$

where the Pauli spinors

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, using (2):

$$U_s\left(\underline{p}\right) = \sqrt{E\left(\underline{p}\right) + m}\begin{pmatrix} \chi_s \\ \dfrac{\underline{\sigma}\cdot\underline{p}}{E\left(\underline{p}\right) + m}\chi_s \end{pmatrix}$$

with normalization

$$U_s\left(\underline{p}\right)^\dagger U_s\left(\underline{p}\right) = (E+m)\left[1 + \frac{\underline{p}^2}{(E+m)^2}\right] = 2E$$

Normalize in a large box, volume V:

$$\int_V \psi^\dagger \psi d^3x = 1$$
$$\rightarrow\ N = \frac{1}{\sqrt{2EV}}$$

and

$$\psi_{p,s}(x)^{\dagger} = \frac{U_s(\underline{p})e^{-ipx}}{\sqrt{2EV}}$$

This describes a particle with energy $E(\underline{p}) > m$, momentum $\underline{p}$, and the component of spin in the $\underline{p}$ direction of $s = \pm \frac{1}{2}$.

In the same way, we can construct 2 negative energy solutions.

It is convenient to write as

$$\psi(x) = Ne^{+ipx} v_s(\underline{p})$$

$e^{+i(Et-\underline{p}x)}$ - energy $E(\underline{p})$, momentum $\underline{p}$.
Note – group velocity $v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$ is in direction of $\underline{p}$.

As before, but eliminating ξ, get (exercise)

$$v_s(\underline{p}) = \sqrt{E(\underline{p}) + m}\begin{pmatrix} \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_{-s} \\ \chi_{-s} \end{pmatrix}$$

and wavefunction

$$\psi_{ps}^{(-)}(x) = v_s(\underline{p})\frac{e^{ipx}}{\sqrt{2EV}}$$

This describes a particle with energy $-E(\underline{p}) < m$, momentum $-\underline{p}$, and the spin component in the $\underline{p}$ direction of $-s = \pm \frac{1}{2}$.

Note that in the non-relativistic limit $|\underline{p}| << m$,

$$U_s(\underline{p}) \to \sqrt{2m}\begin{pmatrix} \chi_s \\ 0 \end{pmatrix}$$

$$v_s(\underline{p}) \to \sqrt{2m}\begin{pmatrix} 0 \\ \chi_{-s} \end{pmatrix}$$

i.e. they reduce to 2-component Pauli spinors.
So – so far:

- No negative densities
- Predicted spin $-\frac{1}{2}$
- Still need + and – energies.