There is propagation if $(E-V)^2 > m^2$.

There are two cases:
Weak potential, $V < E$:
Propagation if $E - V > m$, i.e. $E - m > V$, i.e. if KE is above the barrier, as expected.

Strong potential, $V > E$:
Propagation if $E - V > m$, but can also get propagation if $(E-V) < -m$, i.e. something gets through even at low energy if the barrier is high enough.
What is going on?

Consider the waves on the right, $z > 0$:

$$p'^2 = (V-E)^2 - m^2$$

By $\frac{d}{dp'}$,

$$2p' = 2(V-E)\left(-\frac{dE}{dp'}\right)$$

So the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp'} = -\frac{p'}{V-E},$$

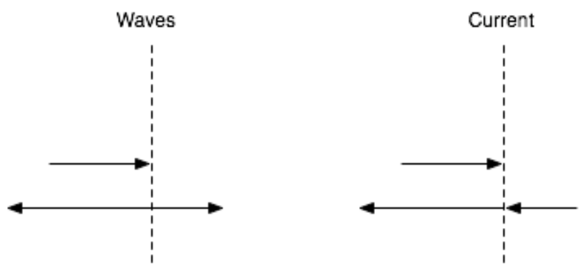
This is greater than 0, so the waves are moving to the right.

Consider the currents. We now have,

$$p' = \sqrt{(V-E)^2 - m^2}.$$

Substituting this into the previous expressions for the currents, we find that for $z < 0$ the reflected current is bigger than the incoming current, $|j_R| > |j_I|$, and for $z > 0$ the transmitted current $j_T < 0$, i.e. the current is negative, and flows to the left.

Summary
For $V > E$, and $V - E > m$, the waves



This suggests an interpretation in terms of anti-particles:
- Particle/antiparticle pairs are created at the barrier if V is big enough.
- Particular to left with reflected waves
- Antiparticles (with opposite charge) go to the right.
- Interpret "conserved charge" as electric charge.

This suggests a similar effect in atoms if Z is large enough. For a full description, we have to abandon single particle theory. Leave this for the moment, and go back to the electron.

3. Dirac Equation

Returning to the start, and trying a different approach.

$$H\psi = i\frac{d\psi}{dt} \quad (1)$$

where

$$H = \sqrt{-\underline{\nabla}^2 + m^2} \quad (2)$$

For the KG equation, we avoided interpreting the square root by using

$$HH\psi = \left(-\underline{\nabla}^2 + m^2\right)\psi = -\frac{\partial^2\psi}{\partial t^2}.$$

This was second order in $\partial / \partial t$, which goes to negative conserved densities. Dirac looked for a 1st order equation of form (1) to describe the electron, with

$$H = -i\underline{\alpha}\cdot\underline{\nabla} + \beta m.$$

i.e.

$$\left(-i\underline{\alpha}\cdot\underline{\nabla} + \beta m\right)\psi = i\frac{\partial\psi}{\partial t} \quad (3)$$

This is the Dirac Equation. The coefficients $\underline{\alpha} = \left(\alpha_1, \alpha_2, \alpha_3\right)$ and β are determined by the requirement that

1. H is hermitian → real E values
   $\rightarrow \alpha, \beta$ are hermitian.
2. The equation
   $$HH\psi = \left(-i\sum_i \alpha_i \frac{\partial}{\partial x_i} + \beta m\right)\left(-i\sum_j \alpha_j \frac{\partial}{\partial x_j} + \beta m\right)\psi = -\frac{\partial^2\psi}{\partial t^2}$$
   is the same as the Klein-Gordon Equation, which will guarantee
   $E^2 = p^2 + m^2$.

Condition 2 is satisfied provided

$$\left\{\alpha_i, \alpha_j\right\} \equiv \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$$

$$\left\{\beta, \alpha_i\right\} = \beta\alpha_i + \alpha_i\beta = 0$$

$$\beta^2 = 1$$

(Collectively equation (4))

[Note: $\{\ \}$ represents the anti-commutator, which uses + rather than -]

β, α can't be numbers – but they can be matrices.

In order to satisfy this, we will need 4 matrices, and we need matrices of order at least 4. For example, (4) are satisfied by:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad (8)$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7)$$

satisfying $\left\{\sigma_i, \sigma_j = 2\delta_{ij}\right\}$.

This solution is called the Dirac representation. There are other choices of 4x4 matrices possible, but they all give the same physics. You don't need a representation really – can instead work from the commutation equations, but this way is convenient.

So the Dirac equation

$$i\frac{\partial\psi}{\partial t} = -i\underline{\alpha}\cdot\underline{\nabla}\psi + \beta m\psi \quad (9)$$

is a matrix equation, as well as a differential equation, with a 4-component wavefunction

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix},$$

which is called a Dirac Spinor.

Since $\underline{\alpha}, \beta$ are hermitian, the adjoint equation (take the complex conjugate) is

$$-i\frac{\partial\psi^*}{\partial t} = +i\underline{\nabla}\cdot(\psi^*\alpha) + m\psi^*\beta, \quad (10)$$

where

$$\psi^\dagger(x) = (\psi_1^*(x), \psi_2^*(x), \psi_3^*(x), \psi_4^*(x)).$$

3.1 Conserved Current

Usual argument: $\psi^\dagger \times (9)$

$$i\psi^\dagger\frac{\partial\psi}{\partial t} - i\psi^\dagger\underline{\alpha}\cdot\underline{\nabla}\psi + \beta m\psi^\dagger\psi \quad \text{(a)}$$

$(10)\times\psi$

$$-i\frac{\partial\psi^\dagger}{dt}\psi = i\underline{\nabla}\psi^\dagger\cdot\alpha\psi + \beta m\psi^\dagger\psi \quad \text{(b)}$$

a – b:

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{j} = 0$$

where $\rho = \psi^\dagger\psi$, $\underline{j} = \psi^\dagger\underline{\alpha}\psi$

In particular,

$$\begin{aligned}\rho &= \psi^\dagger\psi \\ &= \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4 > 0\end{aligned}$$

So this can be treated as a probability density in the usual way. We need to understand the different components.