Relativistic Quantum Mechanics – Lecture 1

So far: $T = E - mc^2 = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

Particles are conserved.
What about relativistic?
Quantum Mechanics + Relativity (Special)
Get some new big ideas:

- Understanding of electron spin, magnetic moment
- Prediction of antiparticles
- Spin-statistics theorem
- Etc…

Eventually leads to quantum field theory QFT.
Take first steps.
Roughly divided into two parts:

1. $T \equiv E - mc^2 < mc^2$ but $T \neq \frac{1}{2}mv^2$
   No particle creation
   Described by "Relativistic Quantum Mechanics"
2. $T = E - mc^2 > mc^2$
   Highly relativistic situation
   Sufficient energy for particle creation
   Need the full works of QFT.

Note: photons are always in region 2. (First QFT was for photons [electromagnetic field].)

In this course – treat (1), and the first steps of (2).

1. Preliminaries
1.1 Relativistic Notation
(Mandl & Shaw, 2.1)
Introduce contravarient 4-vectors.
X^μ, $\mu = 0,1,2,3$
with $\left(X^0, X^1, X^2, X^3\right) = \left(ct, x, y, z\right) = \left(ct, \underline{x}\right)$ (1)
and metric tensor $g_{\mu\nu}$ defined by
$g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$
$g_{\mu\nu} = 0$ if $\mu \neq \nu$. (2)
Define the covariant 4-vector.
$X_\mu = g_{\mu\nu} X^\nu$ with implied sum over repeated indices. (3)
Implies that $X_\mu = \left(ct, -\underline{x}\right)$. (4)
Define the contravarient metric tensor $g^{\mu\nu}$ by $g^{\lambda\mu} g_{\mu\nu} = g^\lambda{}_\nu = \delta^\lambda{}_\nu$ (5)
Where $\delta^\lambda{}_\nu = 1$ if $\lambda = \nu$, 0 otherwise.
Then from $X_\mu = g_{\mu\nu} X^\nu$, we have $g^{\sigma\mu} X_\mu = \underbrace{g^{\sigma\mu} g_{\mu\nu}}_{\delta^\sigma{}_\nu} X^\nu = X^\sigma$, i.e. $X^\sigma = g^{\sigma\mu} X_\mu$. (6)

In general, use $g^{\sigma\mu}$ to raise an index, and $g_{\sigma\nu}$ to lower an index.
From (2) and (6), we have $g^{\lambda\mu} = g_{\lambda\mu}$ (numerically) (7)

<u>Homogenous Lorentz Transformations</u>
Homogenous: $\underline{x}, t = 0 \rightarrow \underline{x}', t' = 0$.
These are of the general form
$X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu$ (8)
where $\Lambda^\mu{}_\nu$ are real and depend on the relative velocity and orientation of the frames.
Together with (3) and (6),
$x_\mu \rightarrow x_\mu' = \Lambda_\mu{}^\nu x_\nu$ (9)
where $\Lambda_\mu{}^\nu = g_{\mu\sigma}\Lambda^\sigma{}_\tau g^{\tau\nu}$ (10)
(Note: $x \equiv X$)

Lorentz transformation leaves the interval $s^2 = x^\mu x_\mu = c^2t^2 - \underline{x}^2$ (11) invariant, i.e.
$X'^\mu X'_\mu = X^\mu X_\mu$.
Using (8) and (9), $X'^\mu X'_\mu = \Lambda^\mu{}_\nu X^\nu \Lambda_\mu{}^\sigma X^\sigma = X^\mu X_\mu$ implies that $\Lambda^\mu{}_\nu \Lambda_\mu{}^\sigma = \delta^\sigma{}_\nu$. (12)
s^2 is an example of a Lorentz scalar.

A 4-component object $A^\mu, B^\mu, ...$ which transforms like X^μ under a Lorentz transformation, e.g. $A^\mu \rightarrow A'^\mu = \Lambda^\mu{}_\nu A^\mu$ (13), is a contravariant 4-vector.
(13) implies $A_\mu = g_{\mu\nu}A^\nu \rightarrow A'_\mu = \Lambda_\mu{}^\nu A_\nu$ (14) (a covariant 4-vector)

The scalar product of any two 4-vectors $AB = A_\mu B^\mu = A^0B_0 - \underline{A}\cdot\underline{B}$ can be written in a variety of ways
$AB = A_\mu B^\mu = g^{\mu\nu}A_\mu B_\nu = g_{\mu\nu}A^\mu B^\nu = A^\mu B_\mu$ (15)
and is Lorentz invariant since $A'^\mu B'_\mu = A^\mu B_\mu$ by (12-14).

Example: the energy-momentum or four-momentum vector.
$$P^\mu = \left(\frac{E}{c}, \underline{p}\right)$$
For single particles, $\frac{E^2}{c^2} - \underline{p}^2 = m^2c^2$ where m is the rest mass.
Or $P^2 = P^\mu P_\mu = m^2c^2$
This is an example of an explicitly covariant equation since $P_\mu P^\mu = P'_\mu P'^\mu$ and hence $P^2 = P'_\mu P'^\mu = m^2c^2$ in a transformed frame.

All the equations written consistently in terms of 4-vectors and scalars (and tensors, but they won't be needed in this course) automatically are Lorentz Invariant.
If the quantities are physical quantities, the equations automatically satisfy the Principle of Special Relativity.