## 8. Mass, Radius and Luminosity Relations

### 8.1 The Eddington Luminosity

Assume radiation pressure.

$$P_{rad} = \frac{1}{3} a T^4$$

$$\frac{dP_{rad}}{dP} = \frac{\kappa \ell}{4\pi c G m(r)} \quad (187)$$

$$P = P_{gas} + P_{rad}$$

$$\frac{dP}{dP} = \frac{dP_{gas}}{dP} + \frac{dP_{rad}}{dP} = 1$$

$$\frac{dP_{rad}}{dP} < 1$$

$$\left( \begin{array}{l} \dfrac{dP_r}{dr} = \dfrac{dP_r}{dP}\dfrac{dR}{dr} \\ \dfrac{dP_{rad}}{dr} = \dfrac{dP}{dT}\dfrac{dT}{dr} = \dfrac{4}{3} a T^3 \dfrac{dT}{dr} \\ \dfrac{dT}{dr} = -\dfrac{3}{4ac}\dfrac{\kappa\rho}{T^3}\dfrac{\ell}{4\pi r^2} \end{array} \right)$$

(Note that all pressures increase inward, therefore all $dP_n / dP$'s are positive).

$$\kappa \ell < 4\pi c G m(r) \quad (188)$$

At the stellar surface,

$$L < \frac{4\pi c G M}{\kappa} \quad (189)$$

For opacity equal to electron scattering,

$$\kappa = \kappa_{es} = const$$

$$L < L_{Ed} = 3.2 \times 10^4 \left( \frac{M}{M_\odot} \right) L_\odot \quad (190)$$

Highest luminosity a stable star can have.

### 8.2 The Mass-Luminosity Relation

The scaling relations are implied by the equations of stellar structure. (The proper derivations use a technique called homology).

Hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -g\rho = -\frac{GM(r)}{r^2}\rho \quad (191)$$

Radiation transport equation:

$$\frac{dT}{dr} = -\frac{\ell(r)}{4\pi r^2}\frac{3}{16}\frac{\kappa\rho}{\sigma T^3} \quad (192)$$

Approximate the density by its average, $\bar{\rho}$.

$$\bar{\rho} \propto \frac{M_*}{R_*^3}$$

Replace the derivative with the ratio $P / R_*$

$$\frac{P}{R_*} = \frac{GM_*}{R_*^2}\frac{M_*}{R_*^3} \propto \frac{M_*^2}{R_*^5}$$

The ideal gas law, $P = \rho kT / \mu$, gives

$$T \propto \frac{P}{\bar{\rho}} \propto \frac{M_*^2}{R_*^4}\frac{R_*^3}{M_*} \propto \frac{M_*}{R_*}$$

Using equation (192),

$$L \propto -R_*^2 T^3 \frac{dT}{dr}\frac{1}{\kappa\rho}$$

Replace derivative with ratio,

$$\frac{dT}{dr} = -\frac{T}{R_*}$$

and we get

$$L \propto \frac{M_*^3}{R_*^3}\frac{R_*^3}{M_*}\frac{M_*}{R_*^2}R_*^2 \propto M_*^3 \quad (193)$$

**8.3 Other scaling relations**

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho$$

$$\frac{P}{R} \sim \frac{M}{R^2}\frac{M}{R^3} \quad (194)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{M}{R} \sim R^2 \rho \quad (195)$$

$$\frac{d\ell}{dr} = 4\pi r^2 \rho\varepsilon$$

$$\frac{L}{R} \sim R^2 \frac{M}{R^3}\varepsilon \quad (196)$$

$$\ell(r) = \frac{4\pi r^2}{\rho}\frac{c}{\kappa}\frac{d}{dr}\left(\frac{aT^4}{3}\right)$$

$$L \sim R^2 \frac{R^3}{M}\frac{1}{\kappa}\frac{T^4}{R} \quad (197)$$

where the luminosity equation assumes radiative energy transport.

We also have some supplementary equations,

$$P_g = \frac{nkT}{\mu}$$
$$P \sim \frac{M}{R^3} T$$
$$P_r = \frac{1}{3} a T^4$$
$$P \sim T^4$$
$$\varepsilon = \varepsilon_0 \rho T^n$$
$$\kappa = \kappa_0 \rho T^{-7/2}$$

As an example, if we assume $\varepsilon = \varepsilon_0 \rho T^{15}$ and constant opacity (i.e. use equation 193),

$$M^{14} \propto R^{18} \quad (198)$$
$$R \propto M^{0.78} \quad (199)$$

For other values of n,

$$R \propto M^{(n-1)/(n+3)} \quad (200)$$

Use this to derive an $\left(L, T_{eff}\right)$ relation.

$$L \propto M^3$$
$$L = 4\pi R^2 {T_{eff}}^4$$

Note: $T_{eff}$ is the surface temperature. $T$ is the core temperature.

$$L^{1-\left(2/3\right)\frac{(n-2)}{(n+3)}} \propto {T_{eff}}^4$$

For $n = 4$ (pp chain)

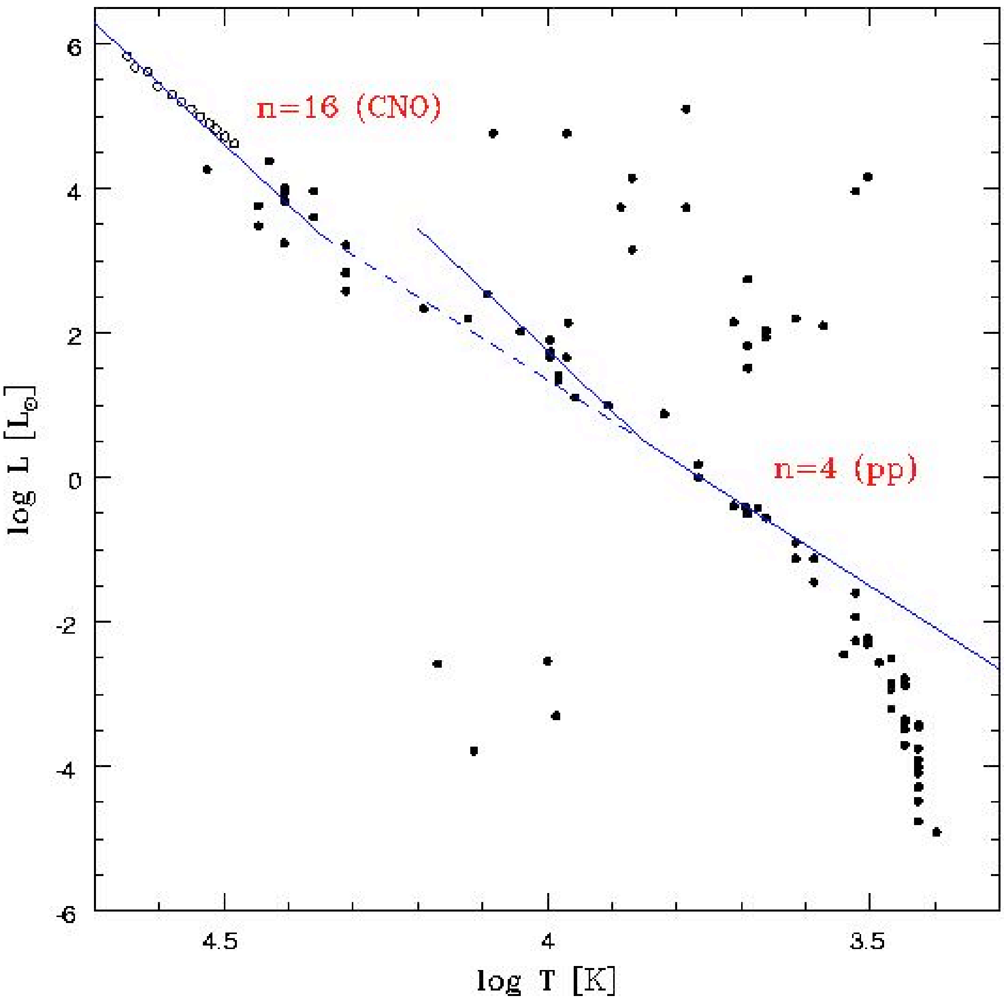
$$\log L = 5.6 \log T_{eff} + const \quad (201)$$

For $n = 16$ (CNO chain)

$$\log L = 8.4 \log T_{eff} + const \quad (202)$$

For fully convective star ($P \propto \rho^{4/3}$ and $n = 4$)

$$\log L = 3.7 \log T_{eff+const} \quad (203)$$




A reasonable fit is obtained to the observed Main Sequence.