6. Limits on the Mass of Stars

6.1 Minimum Mass

Define a star as an object which release energy by nuclear fusion. This sets a minimum value for the central temperature.

To approximate a star, assume a Gaussian model for the pressure gradient. ρ_c is the central density:

$$\frac{dP}{dr} = -\frac{4\pi}{3} C \rho_c^{\,2} r e^{-\frac{r^2}{a^2}} \quad (163)$$

which gives the central pressure P_c as

$$P_c = \left[\frac{\pi}{36}\right]^{1/3} G M^{2/3} \rho_c^{\,4/3} \quad (164)$$

P_c includes gas pressure and electron degeneracy pressure:

$$P_c = K n_e^{\,5/3} + n_i k T_c$$

If the star is pure hydrogen,

$$n_e = n_i = \frac{\rho_c}{m_H}$$

$$K\left(\frac{\rho_c}{m_H}\right)^{5/3} + \frac{\rho_c}{m_H} k T_c = \left[\frac{\pi}{36}\right]^{1/3} G M^{2/3} \rho_c^{\,4/3} \quad (165)$$

$$k T_c = \left[\frac{\pi}{36}\right]^{1/3} G M^{2/3} \rho_c^{\,1/3} - K\left(\frac{\rho_c}{m_H}\right)^{2/3}$$

$$k T_c = A \rho_c^{\,1/3} - B \rho_c^{\,2/3}$$

The maximum occurs when

$$\frac{\partial T_c}{\partial \rho} = \frac{A}{3} \rho_c^{\,-2/3} - \frac{2B}{3} \rho_c^{\,-1/3} = 0$$

$$\rho_c = \left(\frac{A}{2B}\right)^3$$

$$k T_c = \frac{A^2}{4B}$$

The maximum temperature, $T_{\max}$, is given by

$$k T_{\max} = \left[\frac{\pi}{36}\right]^{2/3} \frac{G^2 m_H^{\,8/3}}{4K} M^{4/3}$$

Nuclear fusion may take place if $T_{\max} \geq T_{ign}$ the ignition temperature.
Rearrange to obtain a minimum value for the mass, $M_{\min}$:

$$M_{\min} = \left(\frac{36}{\pi}\right)^{1/2} \left(\frac{4K}{G^2 m_H^{\,8/3}}\right)^{3/4} \left(k T_{ign}\right)^{3/4}$$

where

$$K = \frac{h^2}{20 m_e} \left(\frac{3}{\pi}\right)^{2/3} \quad (166)$$

Substituting for the constants we finally get

$$M_{\min} = 2.79 \times 10^{24} T_{ign}^{3/4} kg \quad (167)$$

For $T_{ign} = 1.5 \times 10^6 k$, $M_{\min} = 0.06 M_{\odot}$

More accurate calculations give $M_{\min} \approx 0.08 M_{\odot}$

(For comparison, $m_{jupiter} \approx 0.001 M_{\odot}$).

6.2 Maximum Mass

Stars are unstable if the pressure is dominated by radiation pressure.

Consider a star with central pressure P_c.

- βP_c is due to gas pressure
- $(1-\beta) P_c$ is due to radiation

$$P_g = \beta P_c = \frac{\rho_c k T}{\mu}$$

$$P_r = \frac{1}{3} \alpha T_c^{\,4} = (1-\beta) P_c$$

Eliminating T gives

$$P_c = \left[\frac{a}{3} \frac{(1-\beta)}{\beta^4} \right]^{1/3} \left[\frac{k \rho_c}{\mu} \right]^{4/3}$$

$$= \left[\frac{\pi}{36} \right]^{1/3} G M_*^{2/3} \rho_c^{4/3} \quad (168)$$

where we assume a Gaussian pressure profile.

Rearranging yields:

$$M_*^{\,2} = \frac{36}{G^3 \pi} \frac{3}{a} \frac{(1-\beta)}{\beta^4} \left[\frac{k}{\mu} \right]^4$$

$$M_* \propto \left[\frac{(1-\beta)}{\beta^4} \right]^{1/2} \quad (169)$$

As M_* increases, β decreases: radiation pressure becomes more important for high-mass stars.

Stability requires gas pressure dominates:

→ $\beta > 0.5$

For $\mu = 0.6 amu$,

$$M(\beta = 0.5) \approx 100 M_{\odot} \quad (170)$$

This is roughly the maximum mass of a star.

For an example of what may happen to a star with a mass around the limit, see Eta Carinae, which was once the second brightest star in the sky.