5. Energy Sources

There are three sources of energy during stellar evolution:

- Gravitational energy
    - Heat due to contraction
    - Pre-main sequence evolution
- Thermonuclear
    - Energy released due to nuclear fusion
    - Main sequence, and post main sequence, evolution.
- Stored energy release
    - Post-nuclear-burning evolution

5.1 Gravitational Energy

Already discussed, the gravitational energy is

$$E_G = -\int_0^{R_*} \frac{Gm(r)}{r} 4\pi r^2 dr$$

which has a typical solution,

$$E_g = -const \times \frac{GM_*^2}{R_*}.$$

For uniform concentration with constant mass:

$$E = const \times GM_*^2 \left[\frac{1}{R_{final}} - \frac{1}{R_{initial}} \right]$$

(it lets itself go. $\rho(r) \to$ contracts; $\rho'(r)$. It can outshine itself in a single nova).

5.2 Thermonuclear Energy

We can write the general equation as

$$X + \alpha \to Z^* \to Y + \beta$$

where X is the target, α the projectile, Z a compound nucleus (* indicating excited state) and Y and β the products of the reaction.

If the mass of the particles is less than the mass of the ingredients, then energy ΔE is released.

$$\Delta E = \Delta mc^2 = \sum_{j,\ ingredients} m_j - \sum_{k,\ products} m_k$$

The energy difference is added to the binding energy of the product.

Consider 4 hydrogen and 1 helium.

$4\,^1H$, $m = 4 \times 1.0081amu = 4.032amu$

$1\,^4He$ $m = 4.0039amu$

i.e. the mass difference is $2.88 \times 10^{-2} amu = 26.5MeV$ (0.7%).

Note on mass / energy units:

$1keV = 1.1605 \times 10^7 k$

$931.1MeV = 1amu$

$1eV = 1.602 \times 10^{-19} J$

Consider $1M_{\odot}$ of Hydrogen being converted to helium. The total mass defect of 0.7% is $1.4\times10^{28}kg = 1.3\times10^{45}J$.
For $L_{\odot} = 4\times10^{26}Js^{-1}$, nuclear reactions will suffice for
$\frac{1.3\times10^{45}}{4\times10^{26}} = 3\times10^{18}s \approx 10^{11}years$.

Nucleus defined by:
- Mass, M_N
- Mass number A, of which
  - Z particles are protons
  - $A-Z$ are neutrons

The binding energy of the nucleus, E_b, is

$$E_b = c^2\left((A-Z)m_n + Zm_p - M_n\right) \quad (123)$$

where m_n, m_p and M_n are the masses of the neutron, proton and nucleus respectively. The binding energy per nucleon is

$$f = \frac{E_b}{A}$$

Note that $H \to He$ is good, but not $He \to Li$.
Nuclear reactions are inhibited by the Coulomb barrier. The potential as a function of distance from the nucleus is.

$$E_{barrier} = \frac{Z_X Z_\alpha e^2}{4\pi\varepsilon_0 r} \quad (124)$$

For hydrogen fusion, $r \sim 10^{-15}m$, so $E_{barrier} \sim 1MeV$. For the sun, the slow energy release suggests that $E_{thermal} << E_{barrier}$. Indeed, at $T = 10^7 k$, $E_{thermal} = kT \sim 1keV << 1MeV$. The average particle has insufficient energy to overcome the barrier. But the particles have a Boltzmann distribution of energies, so what about the high velocity (energy) tail?

At $E >> kT$ the number of particles scales as

$$N \propto e^{-\frac{E-E_{av}}{kT}}$$

For $E \sim 1000kT$, $N \sim e^{-1000} \approx 10^{-434}$
Number of nucleons in the sun, $N \approx 10^{57}$
So the sun isn't hot enough to have nuclear fusion!

Quantum Mechanics comes to the rescue. Particles can tunnel through the barrier (G. Gamov). Wave function decay exponentially through the barrier. Tunneling probability is the amplitude of the wave at the exit point.
Tunneling probability, P_0 for a target of charge $Z_X e$ and a projectile of charge $Z_\alpha e$ is given by

$$P_0 = p_0 E^{-1/2} e^{-2\eta} \quad (125)$$

where

$$\eta = \left(\frac{m}{\ell}\right)^{1/2} \frac{Z_X Z_\alpha e^2}{4\pi\varepsilon_0 \hbar E^{1/2}} \quad (126)$$

where E is the particle energy, and m the reduced mass

$$\frac{1}{m} = \frac{1}{m_x} + \frac{1}{m_\alpha}.$$

P_0 is a function only of the properties of the interacting nuclei. For $T = 10^7 k$, $P_0 \sim 10^{-20 Z_X Z_\alpha}$: at this temperature, only the lowest charge particles can react.

5.3 Nuclear Cross Sections

There are resonant and non resonant reactions. Resonant reactions occur when the projectile has an energy which coincides with an energy state of the compound nucleus. Resonant reactions have a larger cross-section.

5.4 Reaction Rates

$r_{x,\alpha}$ is the number of reactions between species x and α per volume per second.

$$r_{x,\alpha} = n_x n_\alpha \langle \sigma(v) v \rangle \quad (127)$$

where n_x, n_α are the number densities of the target and the projectile respectively.

$\sigma(v)$ is the cross-section for the reaction at a velocity v.

$\langle \sigma(v) v \rangle$ is an average over the velocity distribution.

Use

$$E = \frac{mv^2}{2}$$

where $m = \dfrac{m_x m_\alpha}{m_x + m_\alpha}$ is the reduced mass.

The Maxwell-Boltzmann particle distribution in the energy range E to $E + dE$

$$f(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-\frac{E}{kT}} dE \quad (128)$$

The cross section can be written as (see 126)

$$\sigma(E) = \frac{S(E)}{E} e^{-2\pi\eta}$$

where $S(E)$ hides any non-standard behaviour. We find

$$\langle \sigma(v) v \rangle = \int_0^\infty \sigma(E)\, v\, f(E) dE \quad (129)$$

$$\langle \sigma(v) v \rangle = \frac{2^{3/2}}{(m\pi)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{-\frac{E}{kT} - \frac{\bar{\eta}}{E^{1/2}}} dE \quad (130)$$

where

$$\bar{\eta} = 2\pi\eta E^{1/2} = \frac{\pi (2m)^{1/2} Z_x Z_\alpha e^2}{4\pi\varepsilon_0 \hbar} \quad (131)$$

For non-resonant reactions, take $S(E) = S_0$. The integral in equation 130 becomes:

$$J = \int_0^{\infty} e^{f(E)} dE \quad (132)$$

where

$$f(E) = -\frac{E}{kT} - \frac{\bar{\eta}}{E^{\frac{1}{2}}} \quad (133)$$

The integrand in equation 132 is the product of two exponential functions:

- The *Maxwell-Boltzmann Distribution* which decreases as E increases.
- The *tunneling probability* which increases with E.

The product shows a strong peak, the *Gamow Peak*, at $E = E_0$, where $f(E)$ is maximum.

$$\left(\frac{df(E)}{dE}\right)_{E_0} = 0 \quad (134)$$

Evaluating this derivative, we get:

$$\frac{\bar{\eta}}{2E_0^{\frac{3}{2}}} = \frac{1}{kT} \quad (135)$$

$$E_0 = \left(\frac{\bar{\eta} kT}{2}\right)^{\frac{2}{3}} \quad (136)$$

$$E_0 = \left(\frac{m}{2}\right)^{\frac{1}{2}} \left[\pi \frac{Z_x Z_\alpha e^2 kT}{4\pi\varepsilon_0 \hbar}\right]^{\frac{2}{3}}$$

Introduce a new variable.

$$\tau = 3\frac{E_0}{kT} \quad (137)$$

$$\tau = 3\left(\frac{m}{2kT}\right)^{\frac{1}{3}} \left[\pi \frac{Z_x Z_\alpha e^2}{4\pi\varepsilon_0 \hbar}\right]^{\frac{2}{3}}$$

Substituting this into equation (132) eventually gives

$$J = \int_0^{\infty} e^{-\tau - \frac{\tau}{4}\left(\frac{E}{E_0} - 1\right)^2} dE$$

This is approximately a Gaussian.

Substitute

$$\zeta = \left(\frac{E}{E_0} - 1\right)\frac{\sqrt{\tau}}{2}$$

$$J = \frac{2}{3} kT \tau^{\frac{1}{2}} e^{-\tau} \int_{-\frac{\sqrt{\tau}}{2}}^{\infty} e^{-\zeta^2} d\zeta$$

Most of the contribution to the integral is from close to $E = E_0$: extending the lower limit to $-\infty$ introduces no significant error. (sort of cheating…). The integral then has the value $\sqrt{\pi}$ and so:

$$J = \frac{2}{3} kT \pi^{1/2} \tau^{1/2} e^{-\tau} \quad (138)$$

and the velocity averaged cross section is:

$$\langle \sigma v \rangle = \frac{4}{3} \left(\frac{2}{m} \right)^{1/2} \frac{1}{(kT)^{1/2}} S_0 \tau^{1/2} e^{-\tau} \quad (139)$$

From equation 137:

$$(kT)^{-1/2} \propto \tau^{3/2}$$

so

$$\tau \propto T^{-1/3}$$

Hence

$$\langle \sigma v \rangle \propto \tau^2 e^{-\tau} \quad (140)$$

5.4.1 Width of the Gamow Peak

Peak reaches half maximum at $E = E_0 \pm E_H$, where

$$e^{-\left(\frac{E_0 + E_H}{E_0} - 1\right)^2 \frac{\tau}{4}} = \frac{1}{2}$$

$$\left(\frac{E_H}{E_0} \right)^2 \frac{\tau}{4} = \ln(2)$$

$$\frac{E_H}{E_0} = \frac{2}{\tau^{1/2}} (\ln 2)^{1/2}$$

The width of the peak is $\Delta E = 2E_H$, and so:

$$\frac{\Delta E}{E_0} = \frac{4}{\tau^{1/2}} (\ln 2)^{1/2} \quad (141)$$

5.5 Temperature Dependence of Nuclear Reaction Rates

Write

$$\langle \sigma(v) v \rangle = \langle \sigma(v) v \rangle_0 \left(\frac{T}{T_0} \right)^{\nu} \quad (142)$$

Take log and differentiate.

$$\nu = \frac{\partial \ln \langle \sigma(v) v \rangle}{\partial \ln T} \quad (143)$$

With equation 140, and $\tau \propto T^{-1/3}$:

$$\langle \sigma(v) v \rangle \propto T^{-2/3} e^{-\tau} \quad (144)$$

$$\ln\left(\langle \sigma(v) v \rangle \right) = K - \frac{2}{3} \ln T - \tau$$

$$\frac{\partial \ln \langle \sigma(v) v \rangle}{\partial \ln T} = -\frac{2}{3} - \tau \frac{\partial \ln \tau}{\partial \ln T} = \nu \quad (145)$$

Since

$$\frac{\partial \ln \tau}{\partial \ln T} = -\frac{1}{3}$$

$$\nu = \frac{\partial \ln\langle\sigma(v)v\rangle}{\partial \ln T}$$

$$\rightarrow \nu = \frac{\tau}{3} - \frac{2}{3} \quad (146)$$

Now τ measures the ratio of the energy of the peak of the Gamow peak to the typical thermal energy (equation 137), so $\tau/3 >> 2/3$:

$$\nu \simeq \frac{\tau}{3} \sim T^{-1/3} \quad (147)$$

Write the temperature as $T_7 = \frac{T}{10^7}$. (standard with astronomers. T_a represents units of $\left[10^a k\right]$)

Use the definition of τ to calculate ν for some nuclear reactions. From equation 137 we have

$$\tau = 19.721\left[\left(Z_x Z_\alpha\right)^2 \frac{A_x A_\alpha}{A_x + A_\alpha}\right]^{1/3} T_7^{-1/3}$$

For the proton-proton reaction:

$Z_x = Z_\alpha = 1$

$A_x = A_\alpha = 1$

$\tau_{pp} = 15.6 T_7^{-1/3}$

$\nu_{pp} = 5.2 T_7^{-1/3}$

For the reaction of ^{15}N and p:

$Z_x = 7$

$Z_\alpha = 1$

$A_x = 15$

$A_\alpha = 1$

$\tau_{Np} = 70.6 T_7^{-1/3}$

$\nu_{Np} = 23.5 T_7^{-1/3}$

At $10^7 k$, the reaction rates scale as:

$p - p \;\; r \propto T^{5.2}$

$N - p \;\; r \propto T^{23.5}$!

Carbon is low down here; hence it must be a resonant reaction.

5.6 Nuclear Reactions

Hydrogen burning proceeds via two-body reactions.

pp-1

$p + p \rightarrow {}^2De + e^+ + \nu_e$

${}^2D + p \rightarrow {}^3He + \gamma$

${}^3He + {}^3He \rightarrow {}^4He + 2p \;\; (149)$

(Note that as Deuterium is unstable at these temperatures, the process often stalls at this point (the deuterium falls apart before a proton can hit it). The reaction occurs roughly once every billion years for a particle. This determines the reaction rate of the sun.)

Instead of the last reaction, also

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma \quad (150)$$

This initiates two other chains.

pp-2

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_e$$
$$^{7}Li + p \rightarrow {}^{4}He + {}^{4}He \, (151)$$

pp-III

$$^{7}Be + p \rightarrow {}^{8}B + \gamma$$
$$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_e$$
$$^{8}Be \rightarrow {}^{4}He + {}^{4}He \quad (152)$$

- All three chains can operate simultaneously, but pp-II and pp-III are favoured at higher T .
- The reaction rate is determined by the slowest reaction, $p + p$, with a time scale of $10^{10}\, yrs$
- It acts at the lowest T of any nuclear reaction, and has the weakest T -dependence, $\sim T^{5}$.

At higher T , an alternative process becomes more efficient.

CNO bi-cycle:

$$^{12}C + p \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^{+} + \nu$$
$$^{13}C + p \rightarrow {}^{14}N + \gamma$$
$$^{14}N + p \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_e$$
$$^{15}N + p \rightarrow {}^{12}C + {}^{4}He \quad (153)$$

$$^{15}N + p \rightarrow {}^{16}O + \gamma$$
$$^{16}O + p \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^{+} + \nu_e$$
$$^{17}O + p \rightarrow {}^{14}N + {}^{4}He \quad (154)$$

with $r \propto T^{16}$.

Triple-α:
$^{4}He + {}^{4}He \rightarrow {}^{8}Be$

$^{8}Be + {}^{4}He \rightarrow {}^{12}C \quad (155)$

with $r \propto \rho^{2} T^{40}$
followed by, at low probability
$^{12}C + {}^{4}He \rightarrow {}^{16}O \quad (156)$

C and O burning:

$$\begin{aligned} {}^{12}C + {}^{12}C &\rightarrow {}^{24}Mg + \gamma \\ &\rightarrow {}^{23}Mg + n \\ &\rightarrow {}^{23}Na + p \\ &\rightarrow {}^{20}Ne + {}^{4}He \\ &\rightarrow {}^{16}O + 2\,{}^{4}He \quad (157) \end{aligned}$$

$$\begin{aligned} {}^{16}O + {}^{16}O &\rightarrow {}^{32}S + \gamma \\ &\rightarrow {}^{31}S + n \\ &\rightarrow {}^{31}P + p \\ &\rightarrow {}^{28}Si + {}^{4}He \\ &\rightarrow {}^{24}Mg + 2\,{}^{4}He \quad (158) \end{aligned}$$

Si burning
A series of reactions leading to iron group elements (Fe, Co, Ni).

Photo disintegration:
At very high temperatures, photons can have sufficient energy to dissociate even iron.
$Fe + \gamma \rightarrow 13\,{}^{4}He + 4n \quad (159)$

Neutrinos

Reaction	Chain	Energy
$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_{e}$	pp-1, pp-II, pp-III	$E_{\nu} = 0.263 MeV$
$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_{e}$	pp-II	$0.86, 0.38 MeV$
$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_{e}$	pp-III	$7.2 MeV$
$^{13}N \rightarrow {}^{13}C + e^{+} + \nu$	CNO	$0.71 MeV$
$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_{e}$	CNO	$1.0 MeV$

Neutrino opacity extremely small:
Cross-section for MeV neutrinos $\sim 10^{-48} m^{2}$
Cross-section for photons $\sim 7 \times 10^{-27} m^{2}$

So neutrinos have long mean free paths:

$$\lambda_\nu = \frac{1}{n\sigma_\nu} = \frac{\mu}{\rho\sigma_\nu} = \frac{2\times10^{21}}{\rho} m$$

For $\rho = 10^3 kg\, m^{-3}$

- Photons $\lambda \sim 0.02m$
- But neutrinos $\lambda \sim 2\times10^{18} m \sim 64 pc$

Neutrinos easily escape: their energy is lost and does not heat the Sun.

Energy loss:
$pp-1$ 1%, $pp-II$ 3%, $pp-III$ 27%, CNO 6%

Neutrinos are important because:

1. Effect on energy production
2. Allow observations of the core of the sun

Neutrino Detection Experiments
Measure interactions with neutrinos: large volume of reactants needed.

1. Chlorine
First experiment (1960's) used the reaction
$\nu + {}^{37}Cl \rightarrow e^- + {}^{37}Ar$ (160)
The reaction has an energy threshold of $0.814 MeV$: can not detect neutrinos from ${}^1H + {}^1H$. Mainly sensitive to ${}^8B$ neutrinos (pp-III)

2. Water
Electron scattering reactions.
$\nu + e^- \rightarrow \nu' + e^-$ (161)
Used in Kamiokande. Also sensitive only to higher energy neutrinos, but can measure direction (the direction that the cherenkov radiation is traveling in).

3. Gallium
$\nu + {}^{71}Ga \rightarrow e^- + {}^{71}Ge$ (162)
Sensitive to lower energy neutrinos with $E > 0.23 MeV$

Results:
Only 1/3 of predicted number of neutrinos detected.

- Error in solar models?
- Error in particle physics?

Recent evidence suggests that particle physics is to blame.