3. Equations of State

Equations which describe the physical properties of the material in a star.

3.1 Gas or solid?

Average distance between particles

$$d = \left(\frac{m_N}{\rho}\right)^{1/3}$$

The Coulomb energy per particle is

$$\varepsilon_C \approx \frac{1}{4\pi\varepsilon_0}\frac{Z^2 e^2}{d}$$

The kinetic energy per particle is

$$\varepsilon_k \approx kT \ .$$

It can be estimated from the virial theorem

$$2U + W = 0$$

$$W \approx -\frac{GM^2}{R}$$

$$\varepsilon_k = U\frac{m_N}{M} \approx \frac{GM^2}{2R}\frac{m_N}{M}$$

$$\frac{\varepsilon_C}{\varepsilon_k} \approx \frac{1}{4\pi\varepsilon_0}\frac{Z^2 e^2}{m_N^{4/3} G M^{2/3}} \quad (70)$$

which does not depend on R.

For the sun ($1M_\odot$ of hydrogen), $\varepsilon_C / \varepsilon_k \sim 0.01$.

For a solid, $\varepsilon_C / \varepsilon_k >> 1$

→ stars are gaseous.

3.2 Pressure

What type of gas do we have? Mostly ionized. So it consists of ions and electrons, so we need to write the equation for pressure down twice – one for nuclei, one for electrons. But electrons are far more complicated than this – so the equation only applies for ions.

$$P_{ions} = n_{ions} kT \ .$$

Pressure is momentum transfer in all possible ways. In a star, three components need to be considered:

1. Radiation pressure from photons, P_{rad}
2. Gas pressure
    a. Ion pressure P_I
    b. Electron pressure P_e

$$P = P_{rad} + P_I + P_e \quad (71)$$

This assumes full ionization.

3.3 Photons

The Planck blackbody function $B_\nu(T)$ is related to the energy density in the frequency range ν to $\nu + d\nu$, u_ν by:

$$B_\nu(T) = \frac{c}{4\pi} u_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad (72)$$

The total energy is $E_{rad} = \alpha T^4 Jm^{-3}$ (73)

The pressure of a photon gas is:

$$P_{rad} = \frac{1}{3} E_{rad} = \frac{\alpha T^4}{3} Nm^{-2} \quad (74)$$

3.4 Ideal Monatomic Gas

The well-known results for the pressure and energy density of an ideal gas are:

$$P = nkT\ Nm^{-2} \quad (75)$$

$$E = \frac{3}{2} nkT\ Jm^{-3} \quad (76)$$

Consider a gas consisting of different nuclear species.

$$n_I = \sum_i n_i = \sum_i \frac{\rho}{m_H} \frac{X_i}{A_i} \quad (77)$$

where $X_i = \frac{\rho_i}{\rho}$ is the mass fraction of element i.

The mean atomic mass μ is given by:

$$\frac{1}{\mu_I} = \sum_i \frac{X_i}{A_i} \quad (78)$$

$$n_I = \frac{\rho}{\mu_I m_H}$$

$$P_I = \frac{\rho}{\mu_I m_H} kT \quad (79)$$

Commonly used notation:

$$X_H \to X;$$
$$X_{He} \to Y;$$
$$\sum_{i>2} X_I \to Z$$

(note that this Z is not charge!)

The mean atomic mass can be approximated as:

$$\frac{1}{\mu_I} \approx X + \frac{1}{4} Y + \frac{1 - X - Y}{\langle A_{metals} \rangle} \quad (80)$$

where $\langle A_{metals} \rangle$ is average for $i > 2$.

For young Sun: $X = 0.707$, $Y = 0.274$

$\langle A \rangle \approx 20$ → $\mu_I = 1.29$

Presently in the core of the sun: $X = 0.34$, $Y = 0.64$

→ $\mu_I = 2.0$

3.5 The Saha Equation
How fully is a gas ionized? Consider the balance between photo-ionizations and recombinations, for example Hydrogen:

$$H^+ + e^- \leftrightarrow H^0 + \gamma \;\; \left(E_\gamma = 13.6eV\right) \;\; (81)$$

Forward reaction rate $\propto n_+ n_e$

Backward reaction rate $\propto n_H$

If all particles have a Boltzmann distribution with the same temperature, ionization fraction is set by ratio of rates.

$$\frac{n_+ n_e}{n_0} = \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} e^{-\frac{E_\gamma}{kT}} \;\; (82)$$

which is the Saha equation.

Introduce ionic species, i.e. one for each ionized version of a particle.
The degree of ionization of ionic species i

$$x_i = \frac{n_+}{n_+ + n_0} \;\; (83)$$

where n_0 is prior to electron removal, and n_+ after, for ionic species i.
Electron pressure for ideal gas becomes:

$$P_e = \sum_i x_i \left(n_0 + n_+\right) kT \;\; (84)$$

At high T (i.e. we're fully ionized),

$$\begin{aligned} P_e &\approx \left[\frac{\rho X}{m_H} + \frac{1}{2}\frac{\rho(1-X)}{m_H}\right] kT \\ &\approx \frac{1}{2}(1+X)kT \end{aligned} \;\; (85)$$

3.6 Fermi-Dirac Equation of State
How about a non-ideal gas?
Electrons, protons and neutrons have spin $\pm 1/2$, and are Fermions: every energy state can be occupied by only two particles (2 spin states): Pauli exclusion principle.
3D separation between states:

$$\Delta V \Delta p > h^3 \;\; (86)$$

The number density of particles with momentum p and energy $E(p)$ is:

$$n = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{\frac{E(p)}{kT}} + 1} \;\; (87)$$

Per unit volume for fully occupied states:

$$n_e(p)dp = \frac{2}{\Delta V} = \frac{2}{h^3} 4\pi p^2 dp \;\; \left(p \le p_0\right) \;\; (88)$$

Integrate up to full occupation.

$$n_e = \int_0^{p_0} n_e(p) dp$$

$$p_0 = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3} \;\; (89)$$

If all particles are fully occupied levels, the gas is called fully degenerate.

Define the function

$$F(E) = \frac{1}{e^{\frac{(E - E(p_0))}{kT}} + 1} \quad (90)$$

What happens to $F(E)$ as $T \to 0$?

$T \to 0$, $F(E) \to 0$ if $E > E(p_0)$

$F(E) \to 1$ if $E < E(p_0)$

The transition energy is called the Fermi energy $E_f = E(p_0)$. It is the total energy including the rest mass energy of the most energetic particle.
At $T = 0$, $F(E)$ is a step function, and the number density of particles is given by equation 89.
The pressure is given by:

$$P = \frac{1}{3}\int_0^\infty vpn(p)dp \quad (91)$$

Use

$$n_e(p)dp = \frac{2}{h^3}4\pi p^2 dp \quad (p \le p_0)$$

$$v = \frac{p}{m_e}$$

which gives the degeneracy pressure of the electron gas:

$$\begin{aligned} P_{e,\deg} &= \frac{8\pi}{15 m_e h^3} p_0{}^5 \\ &= \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \end{aligned} \quad (92)$$

(Electrons become degenerate first because of their low mass).
Fill in n_e and find for a non-relativistic case:

$$P_{e,\deg} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1}{2}(1+X)\frac{\rho}{m_H}\right)^{5/3} \quad (93)$$

For relativistic gas, $v \approx c$. Fill in pressure integral, and find:

$$P_{e,\deg}^{rel} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{1}{2}(1+X)\frac{\rho}{m_H}\right)^{4/3} \quad (94)$$

Define

$$\mu_e^{-1} = \frac{1}{2}(1+X)$$

In SI units:

$$P_{e,\deg}^{non-rel} = 1.00 \times 10^7 \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad (95)$$

$$P_{e,\deg}^{rel} = 1.24 \times 10^{10} \left(\frac{\rho}{\mu_e}\right)^{4/3} \quad (96)$$

Summary: Equations of State

$$P = P_{rad} + P_I + P_e \quad (97)$$

$$P_{rad} = \frac{a}{3}T^3 \quad (98)$$

$$P_I = \frac{\rho}{\mu_I m_H}kT \quad (99)$$

$$P_e = \frac{1}{2}(1+X)kT \quad (100)$$

$$P_{e,\deg}^{non-rel} = 1.00\times10^7\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad (101)$$

$$P_{e,\deg}^{rel} = 1.24\times10^{10}\left(\frac{\rho}{\mu_e}\right)^{4/3} \quad (102)$$

At low temperatures, transition from non-relativistic to relativistic degenerate gas occurs at $n_e \sim 10^{35} m^{-3}$.

3.7 Worked Example: White Dwarfs

Consider a cold, fully degenerate gas of electrons $\left(T = 0, P = P_{e,\deg}\right)$.

Assume a degenerate White Dwarf in hydrostatic equilibrium with constant density ρ_w.

A. Calculate its gravitational energy, W.

Recall:

$$dm = 4\pi r^2 \rho_w dr \Rightarrow m(r) = \frac{4\pi}{3}r^3\rho_w$$

$$\begin{aligned} W &= -\int_0^{R_*}\frac{Gm(r)}{r}dm \\ &= -\int_0^{R_*}\frac{G}{r}\frac{4\pi}{3}r^3\rho_w 4\pi r^2\rho_w dr \\ &= -\frac{G(4\pi)^2}{3}\rho_w{}^2\int_0^{R_*}r^4 dr \\ &= -\frac{G(4\pi)^2}{3}\rho_w{}^2\left(\frac{R_*{}^5}{5}\right) \\ &= -\frac{3}{5}\frac{GM_*{}^2}{R_*} \quad (103) \end{aligned}$$

where $M_* = 4\pi R_*{}^3\rho_w$ is the total mass of the star.

B. Calculate the total internal energy U.

The total internal energy is U, which is just equal to the volume times the energy density E, where for non-relativistic gas:

$$E = \int_0^{p_0} n(p)E(p)4\pi p^2 dp \quad (104)$$

… or the easy method:

$$E(p)=\frac{p}{2m}=\frac{3}{2}P \quad (105)$$

$$P=1.00\times10^{7}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

$$U=VE \quad (106)$$

$$=\frac{4\pi}{3}R^{3}\frac{3}{2}\times1.00\times10^{7}\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad (107)$$

In terms of M_* and R_*, we find:

$$U=\frac{1.5\times10^{7}}{\mu_e^{5/3}}\left(\frac{4\pi}{3}\right)^{-2/3}\frac{3M_*}{R_*^{2}} \quad (108)$$

C. Derive a mass-radius relation

In hydrostatic equilibrium, we can apply the Virial Theorem.

$$2U=-W$$

$$K\frac{M^{5/3}}{R_*^{2}}=\frac{M_*^{2}}{R_*} \quad (109)$$

$$KR_*^{-1}=M_*^{1/3}$$

$$M_*\propto R_*^{-3} \quad (110)$$

Adding more mass to a cold, degenerate star makes it smaller!

Substituting all the constants for K above we find the following *radius relation for White Dwarfs*:

$$\frac{M_{wd}}{M_{\odot}}=10^{-6}\left(\frac{R_{wd}}{R_{\odot}}\right)^{-3}\left(\frac{2}{\mu_e}\right)^{5} \quad (111)$$

Example of a white dwarf: Sirius B. This is in a binary star, so we can get its' mass.

Known white dwarfs have masses in the range $0.5\rightarrow1M_{\odot}$, and consist of He or C/O. (from seismic activities of some white dwarfs, we can tell that they have no H in them).

Full ionization implies $\mu_e=2$ (note that μ_e is the mean mass in amu per free electron).

From equation (111), we find:

- $M_{wd}=1M_{\odot}$ → $R_{wd}\approx0.010R_{\odot}$
- $M_{wd}=0.5M_{\odot}$ → $R_{wd}\approx0.0126R_{\odot}$

$0.01R_{\odot}\approx7\times10^{6}m$ is about the radius of the Earth, $6.38\times10^{6}m$

White dwarfs have masses like the Sun, but radii like the Earth.

A "black dwarf" == the Earth…

The same analysis applies to *neutron stars*, containing degenerate neutron gas.

$\mu_e \to \mu_n = 1$ and $m_e \to m_n$.
The mass of a neutron star, M_{ns} is

$$\frac{M_{ns}}{M_\odot} = 5 \times 10^{-15} \left(\frac{R_{ns}}{R_\odot} \right)^{-3} \quad (112)$$

A $1M_\odot$ neutron star has a radius of $11km$!

D. Find the mass limit for white dwarfs
As the density of a degenerate gas increases it eventually becomes fully relativistic, with a different equation of state.
Degenerate electron pressure:

$$P_e = 1.00 \times 10^7 \left(\frac{\rho}{\mu_e} \right)^{5/3} Nm^{-2} \quad (113)$$

Relativistic degenerate pressure;

$$P_e = 1.243 \times 10^{10} \left(\frac{\rho}{\mu_e} \right)^{4/3} Nm^{-2} \quad (114)$$

An ultra relativistic gas has a much higher pressure. Its internal energy density becomes:

$$E = 3P = 3.73 \times 10^{10} \left(\frac{\rho}{\mu_e} \right)^{4/3} \quad (115)$$

and so:

$$\begin{aligned} U &= \frac{4\pi}{3} R_*^3 E \\ &= \frac{4\pi}{3} R_*^3 \frac{1.243 \times 10^{10}}{\mu_e^{4/3}} \left(\frac{3M_*}{4\pi} \right)^{4/3} \frac{1}{R_*^4} \\ &= K' \frac{M_*^{4/3}}{R_*} \quad (116) \end{aligned}$$

The gravitational energy remains unchanged.

$$W \propto \frac{M_*^2}{R_*}$$

Applying the Virial theorem we get:

$$2U = -W$$

$$\frac{M_*^2}{R_*} = K'' \frac{M_*^{4/3}}{R_*} \quad (117)$$

The mass is independent of radius!
More exact calculations than done here give:

$$M_\infty = 1.456 \left(\frac{2}{\mu_e} \right)^2 M_\odot \quad (118)$$

M_∞ is the Chandrasekhar mass – the maximum mass a white dwarf can have.
If more mass than this is present on the star, then the star will collapse.

Neutron stars will be stable, so the white dwarf will go into this form if the mass is too big. Another possibility is that the star will compress until it starts forming carbon, at which point it would explode due to the amount of energy generated.

As there are no neutron stars present with a mass greater than around 2 times that of the sun, it is possible that neutron stars have the same effect – but this hasn't been proven.