5. Dirac Notation

To every state of the system in a state space, we assign a unique symbol. $|\psi\rangle$ (called a KET).
Kets behave like vectors.

Example: the possible states corresponding to the spin of an electron.
Spin up: $|\uparrow\rangle$.
Spin down: $|\downarrow\rangle$.
The state space for the spin of an electron is 2D and $|\uparrow\rangle$, $|\downarrow\rangle$ are a basis.
Then any possible state for the spin of the electron can be written as:
$|\psi_s\rangle = a|\uparrow\rangle + b|\downarrow\rangle$
where $a,b \in \mathbb{C}$.

5.1 Properties of Kets

1) Adding two kets gives another ket.
   $|\psi_1\rangle + |\psi_2\rangle = |\psi_3\rangle$
2) Multiplying a ket by a scalar gives another ket.
   $\alpha|\psi\rangle = |\psi'\rangle$
   where $\alpha \in \mathbb{C}$.
3) The scalar product between $\langle\psi\rangle$ and $|\phi\rangle$ is written as:
   $\langle\psi|\phi\rangle$
4) For every ket $|\psi\rangle$, we assign a “bra” $\langle\psi|$ so that when multiplied with a ket gives the scalar product.
   i.e. $\langle\psi|.|\phi\rangle = \langle\psi|\phi\rangle$
5) The norm of a ket $|\psi\rangle$ is:
   $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$
6) If $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ are an orthonormal basis $\left(\langle\psi_i|\psi_j\rangle = \delta_{ij}\right)$ of the state space. If any $|\phi\rangle$ can be written as:
   $|\phi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle + a_3|\psi_3\rangle$
   $a_1, a_2, a_3 \in \mathbb{C}$
   where $a_1 = \langle\psi_1|\phi\rangle$, $a_2 = \langle\psi_2|\phi\rangle$, $a_3 = \langle\psi_3|\phi\rangle$.
7) We can find the projection of a ket, such as $|\phi\rangle$, onto a subspace by using a projector (projection operator).
   For example, the projector onto the subspace spanned $|\psi_1\rangle$ and $|\psi_2\rangle$ is given as:
   $\widehat{PP}_{12} = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$
   $\widehat{PP}_{12}|\phi\rangle = \left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right)|\phi\rangle$
   $\qquad = \left(|\psi_1\rangle\langle\psi_1|\right)|\phi\rangle + \left(|\psi_2\rangle\langle\psi_2|\right)|\phi\rangle$
   → $\widehat{PP}_{12}|\phi\rangle = |\psi_1\rangle\langle\psi_1|\phi\rangle + |\psi_2\rangle\langle\psi_2|\phi\rangle$

The square of the norm of the projection is:

$$\left\| \widehat{PP}_{12} |\phi\rangle \right\|^2 = |\langle \psi_1|\phi\rangle|^2 + |\langle \psi_2|\phi\rangle|^2$$

8) Using an orthonormal basis $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ an operator $\hat{A}$ can be written as a matrix using:

$$A_{ij} = \langle \psi_i | \hat{A} | \psi_j \rangle$$

9) Important kets:

$$|k\rangle \leftrightarrow \delta(x - x')$$

$$\left[\langle x|\psi\rangle = \int_{-\infty}^{\infty} \delta(x - x')\psi(x')dx' = \psi(x) \right]$$

$$|p_x\rangle \leftrightarrow \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip_x x}{\hbar}}$$

$$\langle p_x|\psi\rangle = \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ip_x x}{\hbar}} \psi(x)dx}_{\text{Fourier Transform}} = \tilde{\psi}(p_x)$$

10) Important Observables:

$$\hat{X}|x\rangle = X|x\rangle$$

$$\hat{p}_x|p_x\rangle = p_x|p_x\rangle$$

i.e. p_x is an eigenvalue, or rather an eigenket of $|p_x\rangle$.

$|1\rangle$, $|2\rangle$, and $|3\rangle$ are an orthonormal basis:

$$|\psi\rangle = a_1|1\rangle + a_2|2\rangle + a_3|3\rangle = \langle 1|\psi\rangle|1\rangle + \langle 2|\psi\rangle|2\rangle + \langle 3|\psi\rangle|3\rangle$$

$$|\psi\rangle = \int a(x)|x\rangle dx = \int \langle x|\psi\rangle|x\rangle dx$$

11) Converting kets into bras (hermitian conjugation)

$$|\psi\rangle \leftrightarrow \langle\psi|$$

$a|\psi\rangle \leftrightarrow a^*\langle\psi|$, $a \in \mathbb{C}$.

$$\hat{A}|\psi\rangle \leftrightarrow \langle\psi|\hat{A}^\dagger$$

$\hat{A}^\dagger$ is the adjoint of $\hat{A}$. If $\hat{A}$ is hermitian, then $\hat{A} = \hat{A}^\dagger$

Example: we want to normalise $|\psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle$ with $|\varphi_1\rangle$ and $|\varphi_2\rangle$ orthonormal.

$$\langle\psi|\psi\rangle = 1$$

$$\langle\psi| = a_1^*\langle\varphi_1| + a_2^*\langle\varphi_2|$$

$$\langle\psi|\psi\rangle = \left(a_1^*\langle\varphi_1| + a_2^*\langle\varphi_2|\right)\left(a_1|\varphi_1\rangle + a_2|\varphi_2\rangle\right)$$

$$= a_1^* a_1 \underbrace{\langle\varphi_1|\varphi_1\rangle}_{1} + a_1^* a_2 \underbrace{\langle\varphi_1|\varphi_2\rangle}_{0} + a_2^* a_1 \underbrace{\langle\varphi_2|\varphi_1\rangle}_{0} + a_2^* a_2 \underbrace{\langle\varphi_2|\varphi_2\rangle}_{1}$$

$$= a_1^* a_1 + a_2^* a_2 = |a_1|^2 + |a_2|^2$$

The normalised ket is:

$$|\psi\rangle = \frac{1}{\sqrt{|a_1|^2 + |a_2|^2}}\left(a_1|\varphi_1\rangle + a_2|\varphi_2\rangle\right)$$

Further reading: Gesorowicz, Chapter 6.