4. Properties of Observables

4.1 Commutation Relations

4.1.1 Definition

The commutator between two observables $\hat{A}$ and $\hat{B}$ is defined by:

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A},$$

where

$$\hat{A}\hat{B}\psi=\hat{A}\left[\hat{B}\psi\right].$$

4.1.2 Properties

- $\left[\hat{A},\hat{B}\right]=-\left[\hat{B},\hat{A}\right]$
- $\left[\hat{A},\hat{A}\right]=0$
- $\left[\hat{A},\hat{B}+\hat{C}\right]=\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]$
- $\left[\hat{A},\hat{B}\hat{C}\right]=\hat{B}\left[\hat{A},\hat{C}\right]+\left[\hat{A},\hat{B}\right]\hat{C}$
- $\left[\hat{A},\hat{A}^2\right]=0$, $\left[\hat{A},\hat{A}^n\right]=0$, $n=1,2,3,...$
- Define $f\left(\hat{A}\right)=\sum_n^{\infty}a_n\hat{A}^n$
- $\left[\hat{A},f\left(\hat{A}\right)\right]=0$

4.1.3 Commuting observables

Also called compatible observables.

We say that two observables $\hat{A}$ and $\hat{B}$ commute if $\left[\hat{A},\hat{B}\right]=0$.

4.1.4 Important property: if $\hat{A}$ and $\hat{B}$ commute

We can find a set orthonormal eigenfunctions common to $\hat{A}$ and $\hat{B}$.

4.1.5 Compatible observables can be measured simultaneously

(seemingly, the title of this section is also the content…)

4.1.6 Examples

1) $\left[\hat{x},\hat{p}_x\right]=?$

$\psi(x)$ is a normalised wavefunction.

$\hat{x}\psi(x)=x\psi(x)$

$\hat{p}_x\psi(x)=-i\hbar\frac{\partial\psi}{\partial x}$

Note that for Hermitian operators, $\int\psi^*\hat{A}\psi=\int\left(\hat{A}\psi\right)^*\psi$

$$\langle[\hat{x},\hat{p}_x]\rangle = \int_{-\infty}^{\infty} \psi^*[\hat{x},\hat{p}_x]\psi(x)dx$$

$$= \int_{-\infty}^{\infty} \psi^*(\hat{x}\hat{p}_x, \hat{p}_x\hat{x})\psi(x)dx$$

$$= \int_{-\infty}^{\infty} \psi^* \hat{x}\hat{p}_x\psi dx - \int_{-\infty}^{\infty} \psi^* \hat{p}_x\hat{x}\psi dx$$

$$= \int_{-\infty}^{\infty} \psi^* \hat{x}(-i\hbar)\frac{\partial\psi}{\partial x}dx - \int_{-\infty}^{\infty} \psi^* \hat{p}_x(x\psi)dx$$

$$= \int_{-\infty}^{\infty} (\hat{x}\psi)^* - i\hbar\frac{\partial\psi}{\partial x}dx - \int_{-\infty}^{\infty} \psi(-i\hbar)\psi dx - \int_{-\infty}^{\infty} \psi(-i\hbar)x\frac{\partial\psi}{\partial x}dx$$

$$= \int_{-\infty}^{\infty} x\psi^*(-i\hbar)\frac{\partial\psi}{\partial x}dx + i\hbar\int_{-\infty}^{\infty} \psi\psi dx + ih\int_{-\infty}^{\infty} \psi x\frac{\partial\psi}{\partial x}dx$$

The first and last terms cancel each other out. $\int_{-\infty}^{\infty} \psi\psi dx = 1$. So

$$\langle[\hat{x},\hat{p}_x]\rangle = i\hbar$$

The result is valid for any ψ. Then:

$[\hat{x},\hat{p}_x]\psi = i\hbar\psi$.

2) In a 4D state space with an orthonormal basis $\varphi_1, \varphi_2, \varphi_3$ and φ_4, an operator $\hat{A}$ has the matrix representation

$$A = \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$$

Eigenvalue	Eigenfunction
$2a$	φ_1
a	φ_2
$-a$	φ_3
$-a$	φ_4

In the same basis the operator $\hat{B}$ is given by

$$B = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & 0 & 2b \\ 0 & 0 & 2b & 0 \end{pmatrix}$$

Eigenvalue	Eigenfunction
$2b$	$\mu_1 = \varphi_1$
b	$\mu_2 = \varphi_2$
$2b$	$\mu_3 = \frac{1}{\sqrt{2}}(\varphi_3 + \varphi_4)$

$-2b$	$\mu_4 = \frac{1}{\sqrt{2}}(\varphi_3 - \varphi_4)$

Do $\hat{A}$ and $\hat{B}$ commute?

$$AB = \begin{pmatrix} 2ab & 0 & 0 & 0 \\ 0 & 2ab & 0 & 0 \\ 0 & 0 & 0 & -2ab \\ 0 & 0 & -2ab & 0 \end{pmatrix} = BA$$

So $AB - BA = 0$ → $\hat{A}$ and $\hat{B}$ commute.

$\hat{A}$		$\hat{B}$	
Eigenvalue	Eigenfunction	Eigenvalue	Eigenfunction
$2a$	φ_1	$2b$	$\mu_1 = \varphi_1$
a	φ_2	b	$\mu_2 = \varphi_2$
$-a$	φ_3	$2b$	$\mu_3 = \frac{1}{\sqrt{2}}(\varphi_3 + \varphi_4)$
$-a$	φ_4	$-2b$	$\mu_4 = \frac{1}{\sqrt{2}}(\varphi_3 - \varphi_4)$

$$\hat{A}\mu_1 = 2a\mu_1$$

$$\hat{A}\mu_2 = a\mu_2$$

$$\hat{A}\mu_3 = \hat{A}\left(\frac{1}{\sqrt{2}}(\varphi_3 + \varphi_4)\right) = \frac{1}{\sqrt{2}}\hat{A}\varphi_3 + \frac{1}{\sqrt{2}}\hat{A}\varphi_4 = \frac{-a}{\sqrt{2}}\varphi_3 - \frac{a}{\sqrt{2}}\varphi_4 = -\frac{a}{\sqrt{2}}(\varphi_3 + \varphi_4) = -a\mu_3$$

$$\hat{A}\mu_4 = \hat{A}\left(\frac{1}{\sqrt{2}}(\varphi_3 - \varphi_4)\right) = \frac{a}{\sqrt{2}}\varphi_3 - \frac{a}{\sqrt{2}}\varphi_4 = -a\mu_4$$

→ μ_1, μ_2, μ_3 and μ_4 are a set of eigenfunctions common to $\hat{A}$ and $\hat{B}$.

We should really rename these μ's so that they are more universally accessible (i.e. anyone will know what μ_1 is, rather than it being an arbitrary choice in this math).

Let:

$$\mu_1 = \mu_{(2a,2b)}$$

$$\mu_2 = \mu_{(a,b)}$$

$$\mu_3 = \mu_{(-a,2b)}$$

$$\mu_4 = \mu_{(-a,-2b)}$$

Evolution in time of expectation values

$\hat{H}$ is independent of time.

System is in a normalised state $\psi(x,t)$.

If we have observable $\hat{A}$, then we want to know how the expectation value acts at different times.

$$\frac{d\langle A\rangle}{dt} = ?$$

$$\langle A\rangle = \langle \psi, \hat{A}\psi\rangle = \int \psi^*(x)\hat{A}\psi(x)dx$$

$$\frac{d\langle A\rangle}{dt} = \left\langle \frac{\partial\psi}{\partial t}, \hat{A}\psi \right\rangle + \left\langle \psi, \frac{\partial\psi}{\partial t} \right\rangle$$

$$ih\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

$$\frac{d\langle A\rangle}{dt} = \left\langle \frac{1}{i\hbar}\hat{H}\psi, \hat{A}\psi \right\rangle + \left\langle \psi, \hat{A}\left(\frac{1}{i\hbar}\right)\hat{H}\psi \right\rangle$$

$$= \left(\frac{1}{ih}\right)\left\langle \hat{H}\psi, \hat{A}\psi \right\rangle + \left(\frac{1}{i\hbar}\right)\left\langle \psi, \hat{A}\hat{H}\psi \right\rangle$$

Where $\hat{H}$ is a Hermitian operator.

$$\left\langle \hat{H}\psi, \psi_2 \right\rangle = \left\langle \psi_1, \hat{H}\psi_2 \right\rangle$$

$$\frac{d\langle A\rangle}{dt} = -\frac{1}{i\hbar}\left\langle \psi, \hat{H}\hat{A}\psi \right\rangle + \frac{1}{i\hbar}\left\langle \psi, \hat{A}\hat{H}\psi \right\rangle$$

$$\left[\hat{A}, \hat{H}\right] = \hat{A}\hat{H} - \hat{H}\hat{A}$$

$$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\left\langle \psi, \left[\hat{A}, \hat{H}\right]\psi \right\rangle$$

$$\rightarrow \frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\left\langle \left[\hat{A}, \hat{H}\right] \right\rangle$$

If $\hat{A}$ and $\hat{H}$ commute (are compatible), then $\langle A\rangle$ is a constant of motion ($\left[\hat{A}, \hat{H}\right] = 0$ therefore $\langle A\rangle = const.$)

If $\hat{A} = \hat{H}$, $\left[\hat{H}, \hat{H}\right] = 0$.

$\frac{d\langle H\rangle}{dt} = 0$ → the energy is a constant of motion.

The eigenvalues of observables that commute with $\hat{H}$ are usually called good quantum numbers.