3) The Postulates of Quantum Mechanics

The State Space

In 3D, each vector corresponds to a point of the physical space. In Quantum Mechanics, each point in the state space (each wave function) corresponds to a possible state of the system. The number of wave functions in a basis of the state space is going to be the number of possible linearly independent states of the system.
For example:
The spin of an electron φ_+ is spin up, while φ_- is spin down. These are the basis functions.
The state space is 2D. So the possible states for the spin of an electron are going to be wavefunctions of the form:
$\psi_3 = a\varphi_+ + b\phi_-$
where $a, b \in \mathbb{C}$.

Postulate 1

All the information about a system is contained in its' wave function ψ, belonging to a state space.
(Using ψ rather than $\psi(x, y, etc)$. $\cos\psi$ could be position in space, momentum space etc. Makes it simpler.)

Postulate 2

To every measurable quantity, there is an associated hermitian operator of observable.

Postulate 3

The only possible result of the measurement of a physical quantitiy is one of the eigenvalues of the corresponding observable

Postulate 4

a) Observables with a discrete spectrum of eigenvalues:
When measuring a physical quantity corresponding to the observable $\hat{A}$ on a system in a normalized state ψ, the probability of the result being equal to the eigenvalue λ_n of $\hat{A}$ will be equal to the square of the norm of the projection of ψ onto the subspace associated to λ_n.

(i) Case λ_n is non-degenerate with Eigenfunction ϕ_n. Then ϕ_n is a basis for the subspace associated to λ_n. The projection of ψ onto this subspace is $\psi_{\lambda_n} = \langle \varphi_n, \psi \rangle \phi_n$. Then the probability of measuring λ_n is
$P(\lambda_n) = \left\| \psi_{\lambda_n} \right\|^2 = \left| \langle \varphi_n, \psi \rangle \right|^2$.

(ii) λ_n is degenerate, with orthonormal eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots, \phi_p$. The projection of ψ onto the subspace associated with λ_n is
$\psi_{\lambda_n} = \sum_i^p \langle \phi_i, \psi \rangle \phi_i$, and the probability is $P(\lambda_n) = \left\| \psi_{\lambda_n} \right\|^2 = \sum_{i=1}^p \left| \langle \phi_i, \psi \rangle \right|^2$.

b) The observable has a continuous non-degenerate spectrum of eigenvalues
When the physical quantity corresponding to $\hat{A}$ is measured on a normalized state ψ,

the probability of obtaining a result between α and $\alpha + d\alpha$ is equal to:
$P(\alpha) = \left|\langle \phi_\alpha, \psi \rangle\right|^2 dx$, where ϕ_α is the eigenfunction associated to the eigenvalue α of $\hat{A}$.

Postulate 5
If the measurement of an observable $\hat{A}$ on a normalized state ψ yields λ_n, then the state of the system immediately after the measurement is the normalized projection of ψ onto the subspace associated with λ_n.

Postulate 6
The time evolution of the wave function $\psi(t)$ is governed by the Schrödinger equation $ih\dfrac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t)$ where (we can postulate) $\hat{H}$ is the observable associated to the total energy of the system (the Hamiltonian).
Example: in a 3D state space, $\varphi_1, \varphi_2, \varphi_3$ form an orthonormal basis and $\hat{A}$ is an observable defined by:
$\hat{A}\varphi_1 = a(\varphi_1 - \varphi_2)$
$\hat{A}\varphi_2 = a(\varphi_2 - \varphi_1)$
$\hat{A}\varphi_3 = 2a\varphi_3$
with a being a real constant.
(i) What values can be obtained from measuring $\hat{A}$?
$A_{ij} = \left\langle \varphi_j, \hat{A}\varphi_j \right\rangle$

$$A_{11} = \left\langle \varphi_1, \hat{A}\varphi_1 \right\rangle = \left\langle \varphi_1, a(\varphi_1 - \varphi_2) \right\rangle = \left\langle \varphi_1, a\varphi_1 \right\rangle + \left\langle \varphi_1, -a\varphi_2 \right\rangle = a\underbrace{\left\langle \varphi_1, \varphi_1 \right\rangle}_{1} - a\underbrace{\left\langle \varphi_1, \varphi_2 \right\rangle}_{0} = a$$

$$A_{12} = \left\langle \varphi_1, \hat{A}\varphi_2 \right\rangle = \left\langle \varphi_1, a(\varphi_2 - \varphi_1) \right\rangle = a\left\langle \varphi_1, \varphi_2 \right\rangle - a\left\langle \varphi_1, \varphi_1 \right\rangle = -a$$

$$A = \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & 2a \end{pmatrix}$$

$$\begin{vmatrix} a-\lambda & -a & 0 \\ -a & a-\lambda & 0 \\ 0 & 0 & 2a-\lambda \end{vmatrix} = 0$$

where λ is the eigenvalue.
$(2a - \lambda)\left[(a-\lambda)^2 - a^2\right] = 0$
$\lambda = 2a$, $\lambda = 0$, $\lambda = 2a$.

$\lambda = 0$:

$$\begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & 2a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$ax - ay = 0$

$-ax+ay=0$
$2az=0$
So:
$z=0$
$x=y$.
So eigenfunctions will be in the form of $(x,x,0)$.
For $\lambda=2a$:

$$\begin{pmatrix} -a & -a & 0 \\ -a & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So we have:
$-ax-ay=0$
$-ax-ay=0$
$0z=0$
So:
$x=-y$
z can be anything.
So eigenfunctions are of the form $(x,-x,z)$. We have the freedom to choose x and z. We choose them by using tha the eigenfunction must be orthogonal.

Eigenvalues	Eigenfunction	Normalised eigenfunctions
2a	$\varphi_1-\varphi_2$	$\frac{1}{\sqrt{2}}(\varphi_1-\varphi_2)=\mu_1$
0	$\varphi_1+\varphi_2$	$\frac{1}{\sqrt{2}}(\varphi_1+\varphi_2)=\mu_2$
2a	φ_3	$\varphi_3=\mu_3$

(ii) A system at time $t=0$ is described by a normalised wave function

$$\psi=\frac{1}{3}\varphi_1-\frac{2i}{3}\varphi_2+\frac{2}{3}\varphi_3$$

If $\hat{A}$ is measured at $t=0$, what results can be found and with what probability? What is the state of the system immediately after the measurement?

If we use postulate 4, then $P(0)=\|\psi_0\|^2$ where ψ_0 s the projection of ψ to the subspace corresponding to 0.

$$\psi_0=\langle\mu_2,\psi\rangle\mu_2$$

$$\left\langle\frac{1}{\sqrt{2}}(\varphi_1+\varphi_2),\frac{1}{3}\varphi_1-\frac{2i}{3}\varphi_2+\frac{2}{3}\varphi_3\right\rangle=\frac{1}{3\sqrt{2}}\langle\varphi_1,\varphi_1\rangle-\frac{2i}{3\sqrt{3}}\langle\varphi_2,\varphi_2\rangle=\frac{1}{3\sqrt{2}}-\frac{\sqrt{2}i}{3}$$

(all the rest are equal to 0. The ones here are equal to 1…)
So:

$$\psi_0=\left(\frac{1}{3\sqrt{2}}-\frac{\sqrt{2}i}{3}\right)\mu_2.$$

$$P(0)=\left|\frac{1}{3\sqrt{2}}-\frac{\sqrt{2}i}{3}\right|^2=\frac{5}{18}$$

Now, find $P(2a) = \|\psi_{2a}\|^2$

$$\psi_{2a} = \langle \mu_1, \psi \rangle \mu_1 + \langle \mu_3, \psi \rangle \mu_3 = \left(\frac{1}{3\sqrt{2}} + i\frac{\sqrt{2}}{3} \right) \mu_1 + \frac{2}{3}\mu_3$$

$$P(2a) = \left| \frac{1}{3\sqrt{2}} + i\frac{\sqrt{2}}{3} \right|^2 + \left| \frac{2}{3} \right|^2 = \frac{13}{18}$$

Immediately after measuring 0, ψ_0 s normalized i.e. $\sqrt{\frac{18}{5}}\psi_0$

Immediately after measuring 2a, the state of the system is $\sqrt{\frac{18}{13}}\psi_{2a}$.

Evolution in time of conservative systems

If the Hamiltonian of a physical system does not depend ot time, the system is said to be conservative.

If $\hat{H}$ has a discrete non-degenerate spectrum of eigenvalues E_n and an orthonormal set of eigenfunctions ϕ_n, then;

$$\hat{H}\phi_n = E_n\phi_n$$

$$i\hbar \frac{\partial \psi_n(x,t)}{\partial t} = \hat{H}\psi_n(x,t)$$

$$\psi_n(x,t) = C_n(t)\phi_n(x)$$

$$i\hbar\phi_n(x)\frac{dC_n(t)}{dt} = C_n(t)\hat{H}\phi_n(x)$$

$$i\hbar\frac{1}{C_n(t)}\frac{dC_n(t)}{dt} = \frac{1}{\phi_n(x)}\hat{H}\phi_n(x)$$

$$i\hbar\frac{1}{C_n(t)}\frac{dC_n(t)}{dt} = E_n$$

Integrating in time between 0 and t;

$$C_n(t) = C_n(0)e^{-\frac{i}{\hbar}E_n t}$$

$$\psi_n(x,t) = C_n(t)\phi_n(t) = C_n(0)e^{-\frac{i}{\hbar}E_n t}\phi_n(x)$$

$$\hat{H}\psi_n(x,t) = C_n(0)e^{-\frac{i}{\hbar}E_n t}\hat{H}\phi_n(x) = E_n C_n(0)e^{-\frac{i}{\hbar}E_n t}\phi_n$$

$$\psi(x,t) = \sum_n A_n\psi_n(x,t)$$

$$\psi(x,t) = \sum_n A_n C_n(0)e^{-\frac{i}{\hbar}E_n t}\phi_n(x)$$